MATH 409 – HOMEWORK 10

Reading: Sections 28, 29, and 30

- **Section 28:** 6 (don't observe, do it), 15, AP1, AP2, AP3
- **Section 29:** 4, 5, 18, AP4
- **Section 30:** 6, 7, AP5

Additional Problem 1:

(a) Use the **definition** of the derivative, as well as the limits from AP3 from last time to show that if $f(x) = \sin(x)$, then $f'(x) = \cos(x)$

(b) Repeat (a) but with $f(x) = \cos(x)$

(c) Use the quotient rule to show that if $f(x) = \tan(x)$, then $f'(x) = \sec^2(x)$

(d) Use the Inverse function theorem to show that if $f(x) = \tan^{-1}(x)$, then $f'(x) = \frac{1}{1+x^2}$ (this is in the notes)

Additional Problem 2: Prove the Quotient Rule by differentiating both sides of the following identity:

$$\ln\left(\frac{f(x)}{g(x)}\right) = \ln\left(f(x)\right) - \ln\left(g(x)\right)$$

Date: Due: Friday, November 19, 2021.

You're allowed to use that $(\ln(x))' = \frac{1}{x}$.

Additional Problem 3:

(a) Consider the following function:

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

Use the definition of the derivative and the hints at the end to show that $f'(0) = 0$

(b) It can be shown (do not do this) that $f^{(n)}(0) = 0$ for all n.

Definition:

The **Maclaurin series** of f is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

What is the Maclaurin series of the function in (a)? Do you think it's a good approximation to f?

Note: This is an example of a **non-analytic** function. In other words, a function which does not equal to its Maclaurin series

Additional Problem 4: Suppose $f(x)$ is a positive function such that $f'(x) = f(x)$ and $f(0) = 1$ (It's true that $f(x) = e^x$ but do **not** use this)

Use the hints at the end to show that for all x and a:

(a) $f(x + a) = f(x)f(a)$

(b) $f(-x) = \frac{1}{f(x)}$

(c) $f(ax) = (f(x))^a$

Congratulations, you've just shown that $e^{x+a} = e^x e^a$, $e^{-x} = \frac{1}{e^x}$, and $e^{ax} = (e^x)^a$ ☺

Additional Problem 5: What happens if you apply L'Hôpital twice to the following limit?

$$\lim_{x \to \infty} \frac{\sqrt{x^2+1}}{x}$$

Find another (calculus) way of evaluating this limit.

Hints:

AP3 Justify all the steps in the following calculation:

$$\frac{1}{x} = x\left(\frac{1}{x}\right)^2 = 2x\frac{1}{2}\left(\frac{1}{x}\right)^2 \leq 2x\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{x}\right)^n = 2xe^{\frac{1}{x}}$$

AP4 For (a), fix a and consider $g(x) = \frac{f(x+a)}{f(x)}$. Then calculate $g'(x)$ and conclude $g(x) = C$. To find C, use $C = g(0)$. For part (b) you consider $g(x) = f(x)f(-x)$ and for part (c) you consider $g(x) = \frac{f(ax)}{f(x)^a}$

28.15 Before you do the inductive step, first show that

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Then use that $(fg)^{(n+1)} = [(fg)^{(n)}]'$. For the sum where you put the derivative on f, it might be good to change variables from $k+1$ to k

and noticing that if k goes from 0 to n, then $k+1$ goes from 1 to $n+1$. Also isolate the two "boundary" terms with $k=0$ and $k=n+1$ so that you can write the middle as a sum from $k=1$ to $k=n$. This allows you to use the formula above.

29.18 First prove by induction that $|s_{n+1}-s_n| \leq a^n |s_1 - s_0|$. Then justify the steps in the following calculation (assume WLOG $n > m$)

$$\begin{aligned} |s_n - s_m| &\leq |s_n - s_{n-1}| + \cdots + |s_{m+1} - s_m| \\ &\leq a^{n-1} |s_1 - s_0| + \cdots + a^m |s_1 - s_0| \\ &\leq a^m |s_1 - s_0| \sum_{k=0}^{m-n-1} a^k \\ &\leq a^m |s_1 - s_0| \frac{1}{1-a} \end{aligned}$$

Finally, since $\lim_{m\to\infty} a^m = 0$ (since $a < 1$), there is N such that if $m > N$ then $a^m < \frac{\epsilon(1-a)}{|s_1 - s_0|}$ and use this to conclude that (s_n) is Cauchy.