HOMEWORK 11 – AP SOLUTIONS

AP 1

Here $n = 2$ and $a_1 = 0$ and $A_1 =$ rationals in $[0, 1]$, which has measure $m(A_1) = 0$, and $a_2 = 1$ and $A_2 =$ irrationals in $[0, 1]$, which has measure $m(A_2) = 1 - m(A_1) = 1 - 0 = 1$, and therefore by the definition, we have

$$\int_0^1 f(x)dx = a_1 m(A_1) + a_2 m(A_2) = 0 \times 0 + 1 \times 1 = 1$$

AP 2

Let P be the partition $P = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$

Since $f(x) = x$ is increasing, then

$$M(f, [t_{k-1}, t_k]) = f(t_k) = t_k \text{ (Right Endpoint)}$$

$$\begin{aligned} U(f, P) &= \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) \left(\alpha(t_k) - \alpha(t_{k-1})\right) \\ &= \sum_{k=1}^{n} t_k \left((t_k)^2 - (t_{k-1})^2\right) \end{aligned}$$

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Given n, let P be the evenly spaced Calculus partition with $t_k = \frac{k}{n}$:

In that case,

$$(t_k)^2 - (t_{k-1})^2 = \frac{k^2}{n^2} - \frac{(k-1)^2}{n^2} = \frac{k^2 - k^2 + 2k - 1}{n^2} = \frac{2k-1}{n^2}$$

$$\begin{aligned}
U(f,P) &= \sum_{k=1}^{n} \left(\frac{k}{n}\right)\left(\frac{2k-1}{n^2}\right) \\
&= \sum_{k=1}^{n} \frac{2k^2 - k}{n^3} \\
&= \frac{2}{n^3}\sum_{k=1}^{n} k^2 - \frac{1}{n^3}\sum_{k=1}^{n} k \\
&= \frac{2}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) - \frac{1}{n^3}\frac{n(n+1)}{2} \\
&= \frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{2n^2}
\end{aligned}$$

Since $U(f)$ is the inf over all partitions, we must have

$$U(f) \leq U(f,P) = \frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{2n^2}$$

Therefore, taking the limit as $n \to \infty$ of the right hand side, we get $U(f) \leq \frac{2}{3} - 0 = \frac{2}{3}$

This is similar to the above, except that here $m(f, [t_{k-1}, t_k]) = t_{k-1}$ (Left endpoint), and so, using sup we get $L(f) \geq \frac{2}{3}$.

Since $U(f) \leq \frac{2}{3} \leq L(f)$ and because $L(f) \leq U(f)$, we get $L(f) = U(f) = \frac{2}{3}$.

Hence $\int_0^1 x d\alpha(x) = \frac{2}{3}$.

Note: This is actually the same as:

$$\int_0^1 x\alpha'(x)dx = \int_0^1 x(2x)dx = 2\int_0^1 x^2 dx = \frac{2}{3}$$

AP 3

$$\begin{aligned}
\ln\left(\prod_a^b (f(x))^{dx}\right) &= \ln\left(\lim_{n\to\infty} (f(x_1))^{t_1-t_0} (f(x_2))^{t_2-t_1} \cdots (f(x_n))^{t_n-t_{n-1}}\right) \\
&= \lim_{n\to\infty} \ln\left((f(x_1))^{t_1-t_0} (f(x_2))^{t_2-t_1} \cdots (f(x_n))^{t_n-t_{n-1}}\right) \\
&= \lim_{n\to\infty} (t_1-t_0)\ln(f(x_1)) + (t_2-t_1)\ln(f(x_2)) \\
&+ \cdots + (t_n-t_{n-1})\ln(f(x_n)) \\
&= \lim_{n\to\infty} \sum_{k=1}^n \ln(f(x_k))(t_k-t_{k-1}) \\
&= \int_a^b \ln(f(x))\,dx
\end{aligned}$$

$$\text{Therefore: } \prod_a^b (f(x))^{dx} = e^{\int_a^b \ln(f(x))dx}$$

AP 4(a)

First suppose $p \neq 1$

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x^p}dx \\ &= \lim_{t\to\infty}\int_1^t x^{-p}dx \\ &= \lim_{t\to\infty}\left[\frac{x^{1-p}}{1-p}\right]_{x=1}^{x=t} \\ &= \lim_{t\to\infty}\frac{t^{1-p}}{1-p}-\frac{1}{1-p}\end{aligned}$$

But if $p<1$, then $1-p>0$ and so $t^{1-p}\to\infty$, and so the integral diverges. And if $p>1$, then $1-p<0$, and so $t^{1-p}\to 0$ and so the integral converges to $\frac{-1}{1-p}=\frac{1}{p-1}$

Finally, if $p=1$, then

$$\begin{aligned}\int_1^\infty \frac{1}{x}dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x}dx \\ &= \lim_{t\to\infty}\ln(t)-\ln(1) \\ &= \lim_{t\to\infty}\ln(t) \\ &= \infty\end{aligned}$$

And so if $p=1$, the integral diverges as well.

Therefore, the integral converges if and only if $p>1$

AP 4(B)

The (possible) singularity here is at $x=0$. Again first suppose $p\neq 1$

$$\begin{aligned}\int_0^1 \frac{1}{x^p}dx &= \lim_{t\to 0^+}\int_t^1 \frac{1}{x^p}dx\\ &= \lim_{t\to 0^+}\int_t^1 x^{-p}dx\\ &= \lim_{t\to 0^+}\left[\frac{x^{1-p}}{1-p}\right]_{x=t}^{x=1}\\ &= \lim_{t\to 0^+}\frac{1}{1-p}-\frac{t^{1-p}}{1-p}\end{aligned}$$

But if $p < 1$, then $1-p > 0$ and so $t^{1-p} \to 0$, and so the integral converges to $\frac{1}{1-p}$. And if $p > 1$, then $1-p < 0$, and so $t^{1-p} \to \infty$ and so the integral diverges

Finally, if $p = 1$, then

$$\begin{aligned}\int_0^1 \frac{1}{x}dx &= \lim_{t\to 0^+}\int_t^1 \frac{1}{x}dx\\ &= \lim_{t\to 0^+}\ln(1)-\ln(t)\\ &= \lim_{t\to 0^+}-\ln(t)\\ &=\infty\end{aligned}$$

And so if $p = 1$, the integral diverges as well.

Therefore, the integral converges if and only if $p < 1$

AP 5

Notice that our integral equation

$$\left(\int_0^x f(t)dt\right)\left(\int_0^x \frac{1}{f(t)}dt\right) = x^2$$

Becomes:

$$\left(\int_0^x \left(\sqrt{f}\right)^2 dt\right)\left(\int_0^x \left(\frac{1}{\sqrt{f}}\right)^2 dt\right) = \left(\int_0^x 1dt\right)^2$$

And since $1 = \sqrt{f}\left(\frac{1}{\sqrt{f}}\right)$, we have equality in the Cauchy-Schwarz inequality with $\sqrt{f}$ and $\frac{1}{\sqrt{f}}$.

Therefore one function is a multiple of the other one, say $\sqrt{f} = C\left(\frac{1}{\sqrt{f}}\right)$ and cross-multiplying we get $\left(\sqrt{f}\right)^2 = C$ and so $f = C$ (here $C > 0$ since f is positive), and you can check that $f = C$ satisfies the original equation as well. □