HOMEWORK 11 – SELECTED BOOK SOLUTIONS

32.1

STEP 1: Partition

$$P = \{0 = t_0 < t_1 < \cdots < t_n = b\}$$

STEP 2: $U(f, P)$

Since x^3 is increasing, we have

$$M(f, [t_{k-1}, t_k]) = f(t_k) = (t_k)^3 \quad \text{(Right Endpoint)}$$

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) (t_k - t_{k-1}) = \sum_{k=1}^{n} (t_k)^3 (t_k - t_{k-1})$$

STEP 3: $U(f)$

Given n, let P be the evenly spaced Calculus partition with $t_k = \frac{bk}{n}$

In that case $t_k - t_{k-1} = \frac{b}{n}$ and

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$$\begin{aligned}
U(f,P) &= \sum_{k=1}^{n} \left(\frac{bk}{n}\right)^3 \left(\frac{b}{n}\right) \\
&= \sum_{k=1}^{n} \frac{b^4 k^3}{n^4} \\
&= \frac{b^4}{n^4} \sum_{k=1}^{n} k^3 \\
&= \frac{b^4}{n^4} \left(\frac{n^2(n+1)^2}{4}\right) \\
&= \frac{b^4(n+1)^2}{4n^2} \\
&= \frac{b^4}{4} \left(\frac{n+1}{n}\right)^2
\end{aligned}$$

Upshot: Since $U(f)$ is the inf over all partitions, we must have

$$U(f) \leq U(f,P) = \frac{b^4}{4}\left(\frac{n+1}{n}\right)^2$$

Therefore, taking the limit as $n \to \infty$ of the right hand side, we get $U(f) \leq \frac{b^4}{4}$

STEP 4: $L(f)$

This is similar to the above, except that here $m(f, [t_{k-1}, t_k]) = (t_{k-1})^3$ (Left endpoint), and so, using sup we get $L(f) \geq \frac{b^4}{4}$.

Since $U(f) \leq \frac{b^4}{4} \leq L(f)$ and because $L(f) \leq U(f)$, we get $L(f) = U(f) = \frac{b^4}{4}$. Hence $f(x) = x^3$ is Darboux integrable and $\int_0^b x^3 dx = \frac{b^4}{4}$.

33.4

Consider the following function f on $[0,1]$

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

Then for any partition P, $M(f,[t_{k-1},t_k]) = 1$ and $m(f,[t_{k-1},t_k]) = -1$, and so

$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} 1(t_k - t_{k-1}) = t_n - t_0 = 1 - 0 = 1$$

And so, taking inf over all partitions P, we get $U(f) = 1$

$$L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n}(-1)(t_k - t_{k-1}) = -(t_n - t_0) = -1$$

And taking sup over all partitions P, we get $L(f) = -1$.

Since $U(f) \neq L(f)$, f is not Darboux integrable

However, $|f| = 1$ (the constant function 1), which is integrable

33.7

Fix a partition P, then for any x and y in a given sub-piece $[t_{k-1},t_k]$, we have

$$\begin{aligned}
(f(x))^2 - (f(y))^2 &= (f(x) + f(y))\,(f(x) - f(y)) \\
&\leq |f(x) + f(y)|\,|f(x) - f(y)| \\
&\leq (|f(x)| + |f(y)|)\,|f(x) - f(y)| \\
&\leq (B + B)\,|f(x) - f(y)| \\
&= 2B\,|f(x) - f(y)|
\end{aligned}$$

Then, taking the sup over $x \in [t_{k-1}, t_k]$ and then the inf over $y \in [t_{k-1}, t_k]$, we get

$$\begin{aligned}
M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k]) &\leq 2B\,|M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])| \\
&= 2B\,(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]))
\end{aligned}$$

(Here we used the fact that $M \geq m$)

Finally, summing over k, we get

$$U(f^2, P) - L(f^2, P) \leq 2B\,(U(f, P) - L(f, P))$$

For part (b), let $\epsilon > 0$ be given, then since f is integrable on $[a, b]$, by the Cauchy Criterion, there is a partition P such that $U(f, P) - L(f, P) < \frac{\epsilon}{2B}$.

With the same P, using the result of (a), we get

$$U(f^2, P) - L(f^2, P) \leq 2B\,(U(f, P) - L(f, P)) < (2B)\left(\frac{\epsilon}{2B}\right) = \epsilon \checkmark$$

Hence, by the Cauchy criterion again, f^2 is integrable on $[a, b]$

33.8

Since f and g are integrable on $[a, b]$, then so are $f+g$ and $f-g$. Then since $f+g$ and $f-g$ are bounded (since f and g are), by the previous exercise, $(f+g)^2$ and $(f-g)^2$ are integrable on $[a, b]$, and therefore so is

$$fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right)$$

For part (b), since $f-g$ is integrable on $[a, b]$, so is $|f-g|$, and therefore

$$\max(f, g) = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$$

is integrable on $[a, b]$. Similarly,

$$\min(f, g) = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$$

is integrable on $[a, b]$

33.10

Let $\epsilon > 0$ be given, then since $f(x)$ is continuous on $[\frac{\epsilon}{8}, 1]$, by the Cauchy criterion, there is a partition P_1 on that interval such that $U(f, P_1) - L(f, P_1) < \frac{\epsilon}{4}$

Similarly there is a partition P_2 on $[-1, -\frac{\epsilon}{8}]$ such that $U(f, P_2) - L(f, P_2) < \frac{\epsilon}{4}$.

Let $P = P_1 \cup P_2$, which is a partition of $[-1, 1]$

Then since

$$M(f, [-\frac{\epsilon}{8}, \frac{\epsilon}{8}]) - m(f, [-\frac{\epsilon}{8}, \frac{\epsilon}{8}]) \left(\frac{\epsilon}{8} - \frac{-\epsilon}{8}\right) < (1 - (-1))\frac{\epsilon}{4} = \frac{\epsilon}{2}$$

We ultimately obtain $U(f,P) - L(f,P) < \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4} = \epsilon$ and therefore by the Cauchy criterion, f is integrable on $[-1,1]$

34.8

Let $u = \tan^{-1}(x)$, $dv = x$, then $du = \frac{1}{x^2+1}$, $v = \frac{x^2}{2}$, and so

$$\begin{aligned}
\int_0^1 x\tan^{-1}(x)dx &= \left[\frac{x^2}{2}\tan^{-1}(x)\right]_0^1 - \int_0^1 \left(\frac{x^2}{2}\right)\left(\frac{1}{x^2+1}\right)dx \\
&= \frac{1}{2}\tan^{-1}(1) - \frac{1}{2}\int_0^1 \frac{x^2}{x^2+1}dx \\
&= \frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{2}\int_0^1 1 - \frac{1}{x^2+1}dx \\
&= \frac{\pi}{8} - \frac{1}{2}\left[x - \tan^{-1}(x)\right]_0^1 \\
&= \frac{\pi}{8} - \frac{1}{2}\left(1 - \tan^{-1}(1) - 0 + \tan^{-1}(0)\right) \\
&= \frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right) \\
&= \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} \\
&= \frac{\pi}{4} - \frac{1}{2}
\end{aligned}$$

How painful! ☹

This time let $u = \tan^{-1}(x)$, $dv = x$ but $du = \frac{1}{x^2+1}$ $v = \frac{x^2+1}{2}$, and so

$$\begin{aligned}\int_0^1 x\tan^{-1}(x)dx =& \left[\frac{x^2+1}{2}\tan^{-1}(x)\right]_0^1 - \int_0^1 \left(\frac{x^2+1}{2}\right)\left(\frac{1}{x^2+1}\right)dx \\ =& \frac{2}{2}\tan^{-1}(1) - \int_0^1 \frac{1}{2}dx \\ =& \left(\frac{\pi}{4}\right) - \frac{1}{2}\end{aligned}$$

Soooo much better! ☺

As for the second integral, let $u = \ln(x+2)$, $dv = 1$, then $du = \frac{1}{x+2}$, $v = x+2$, and so

$$\int \ln(x+2)dx = (x+2)\ln(x+2) - \int (x+2)\left(\frac{1}{x+2}\right)dx = (x+2)\ln(x+2) - x + C$$

Finally, for the third integral, let $u = \tan^{-1}\left(\sqrt{x+1}\right)$, $dv = 1$, then $du = \frac{1}{1+\left(\sqrt{x+1}\right)^2}\left(\frac{1}{2\sqrt{x+1}}\right) = \frac{1}{x+2}\left(\frac{1}{2\sqrt{x+1}}\right)$ and $dv = x+2$ and so

$$\begin{aligned}\int \tan^{-1}\left(\sqrt{x+1}\right)dx =& (x+2)\tan^{-1}\left(\sqrt{x+1}\right) - \int (x+2)\left(\frac{1}{x+2}\right)\left(\frac{1}{2\sqrt{x+1}}\right)dx \\ =& (x+2)\tan^{-1}\left(\sqrt{x+1}\right) - \int \frac{1}{2\sqrt{x+1}}dx \\ =& (x+2)\tan^{-1}\left(\sqrt{x+1}\right) - \sqrt{x+1} + C\end{aligned}$$

34.10

$\int_0^1 g(x)dx$ is the area under the graph of $y = g(x)$, whereas $\int_0^1 g^{-1}(u)du$ is the area to the *left* of the graph of $y = g(x)$ (since $y = g(x) \Leftrightarrow x = g^{-1}(y)$). Therefore the sum of the integrals is just the area of the square with sides $[0,1]$ and $[0,1]$, which is 1