MATH 409 – HOMEWORK 11

Reading: Sections 32 and 33. You don't need to read section 34 to do the problems below

- **Section 32:** 1, AP1, AP2
- **Section 33:** 4, 7, 8, 10, AP3
- **Section 34:** 8 (See Note), 10, AP4, AP5

Note: Also apply the same trick from 8(b) to find $\int \ln(x+2)dx$ and $\int \tan^{-1}\left(\sqrt{x+1}\right)dx$ (here choose $x+2$ instead of $x+1$)

Additional Problem 1: Consider the following function $f(x)$ on $[0,1]$

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

We've seen that f is not Darboux integrable. Here is a better way of finding $\int_0^1 f(x)dx$ called the **Lebesgue Integral**:

Definition:

If f is a function having values $a_1, a_2, \cdots, a_n$ on sets $A_1, A_2, \cdots, A_n$ respectively, then the **Lebesgue integral** of f is:

$$\int f(x)dx = a_1 m(A_1) + a_2 m(A_2) + \cdots + a_n m(A_n)$$

Date: Due: Friday, December 3, 2021.

Here m stands for measure or size. Using this definition, evaluate $\int_0^1 f(x)dx$ with f as above. You may use the fact that the measure of the rational numbers in $[0,1]$ is 0

Hint: Here f has only two values, so $n = 2$.

Additional Problem 2: We may generalize the Darboux integral to the **Riemann-Stieltjes Integral** $\int_a^b f(x)d\alpha(x)$.

The *only* difference between this and the Darboux integral is that, in $U(f,P)$ and $L(f,P)$, we use $\alpha(t_k) - \alpha(t_{k-1})$ instead of $t_k - t_{k-1}$

With this definition, and mimicking Example 1 in section 32, evaluate $\int_0^1 xd(x^2)$, so here $\alpha(x) = x^2$.

Additional Problem 3: The **Product integral** $\prod_a^b f(x)dx$ is defined as follows:

$$\prod_a^b (f(x))^{dx} = \lim_{n\to\infty} (f(x_1))^{t_1-t_0} (f(x_2))^{t_2-t_1} \cdots (f(x_n))^{t_n-t_{n-1}}$$

Where x_k is a random point in $[t_{k-1}, t_k]$

What is $\prod_a^b f(x)^{dx}$ in terms of the Riemann integral $\int_a^b f(x)dx$?

Here you're exceptionally allowed to be nonrigorous and put functions inside limits ☺. I just want a formula here

Hint: How do you turn a product into a sum?

Additional Problem 4: We can extend the notion of integrals to evaluate improper integrals

Definition:

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

Definition:

If f has an infinite discontinuity at a, then

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$$

Use those definitions, show that:

(a) $\int_1^\infty \frac{1}{x^p}dx < \infty \Leftrightarrow p > 1$

(b) $\int_0^1 \frac{1}{x^p}dx < \infty \Leftrightarrow p < 1$

Additional Problem 5: We can generalize the Cauchy Schwarz inequality to integrals:

Cauchy-Schwarz for Integrals: (don't prove)

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \left(\int_a^b (f(x))^2\,dx\right)\left(\int_a^b (g(x))^2\,dx\right)$$

With equality iff one function is a multiple of the other

Use the above to show that the only positive solutions to the following equation are the constant functions $f(x) = C$ (with $C > 0$):

$$\left(\int_0^x f(t)dt\right)\left(\int_0^x \frac{1}{f(t)}dt\right) = x^2$$

Hint: $f = \left(\sqrt{f}\right)^2$ and $\frac{1}{f} = \left(\frac{1}{\sqrt{f}}\right)^2$ and $x = \int_0^x 1dt$