MATH 409 – HOMEWORK 2

Readings: Sections 2, 3, and 4. Ignore the Rational Zeros Theorem in section 2; it's not very important for this class. No need to memorize the field/ordered field axioms in section 3, but know how to prove all the properties in that section. The Triangle Inequality is **very** important. Section 4 is **very** hard but **the** most important section in the book, make sure to understand every single detail in it. The Archimedean Property and the Denseness of $\mathbb{Q}$ will be part of the next homework.

- **Section 2:** 4, AP1 (Optional: AP5)
- **Section 3:** 5, 6b, 8, AP2, AP3, (Optional: AP6)
- **Section 4:** 7, 8, 14, AP4

Hints: For 4 in section 2, just do it by contradiction; you may assume $\sqrt{3}$ is irrational. For 8 in section 3, do it by contradiction.

Additional Problem 1: Here is how to actually construct $\mathbb{Q}$: Let $A = \mathbb{Z} \times \mathbb{Z}^{\star}$ (where $\mathbb{Z}^{\star}$ is the set of nonzero integers) and define a relation $\sim$ on A by:

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc$$

(a) Prove that $\sim$ is an equivalence relation on A (see this link for a definition)

Date: Due: Friday, September 10, 2021.

(b) Give three examples of elements in the equivalence class of $[(1,2)]$. This equivalence class is what we call $\frac{1}{2}$.

Additional Problem 2: Here are some applications of the triangle inequality in real (analysis) life.

(a) Show that if $|x-y| < \frac{\epsilon}{2}$ and $|y| < \frac{\epsilon}{2}$, then $|x| < \epsilon$

(b) Show that if $|x-y| < \frac{\epsilon}{3}$ and $|y-z| < \frac{\epsilon}{3}$ and $|z-t| < \frac{\epsilon}{3}$, then $|x-t| < \epsilon$

(c) Show if $|x-z| \leq \frac{\epsilon}{2}$ and $|y-z| \geq \epsilon$, then $|y-x| \geq \frac{1}{2}|y-z|$.

Hint for (c)**:** Start with $|y-z|$ and write $y-z$ in terms of $y-x$ and $x-z$, and then use $\epsilon \leq |y-z|$.

Additional Problem 3: Use the definition of sup and inf to find the following. Explain why your answer is correct.

(a) $\sup(A)$ where $A = \left\{3 - \frac{2}{n}, n \in \mathbb{N}\right\}$

(b) $\inf(B)$ where $B = \{e^{-x}, x \in \mathbb{R}\}$

(c) $\sup(C)$ where $C = \{n(-1)^n, n \in \mathbb{N}\}$

Hint: Is C bounded above? Why or why not?

Additional Problem 4: Let A and B be nonempty bounded subsets of $\mathbb{R}$ and let AB be the set of all products ab where $a \in A$ and $b \in B$. Is it always true that $\sup(AB) = \sup(A)\sup(B)$? Why or why not?

Optional Additional Problem 5 Prove that the set of algebraic numbers are countable, as follows. You may use (without proof) the fundamental theorem of algebra, which says that every polynomial of degree $n \geq 1$ with integer coefficients (not all 0) has at most n roots.

(a) First, show that for **fixed** $n \geq 1$ and **fixed** integers $a_n, \cdots, a_1, a_0$ (not al 0), the set of solutions to $a_n x^n + \cdots + a_1 x + a_0 = 0$ is countable

(b) Then show that for **fixed** n, the set of zeros of all polynomials of degree n with integer coefficients (not all 0) is countable

(c) Finally show that the set of algebraic numbers are countable

Optional Additional Problem 6: Show that there is no order structure $\leq$ on $\mathbb{C}$ that agrees with the standard order structure $\leq$ on $\mathbb{R}$.

Hint: Consider two cases: $i \geq 0$ and $i \leq 0$.

Note: In fact, more is true! There is no order structure on $\mathbb{C}$ at all! See this video if you're interested: Can you compare complex numbers?