HOMEWORK 2 – AP SOLUTIONS

AP 1:

(a) **Reflexive:** Do we have $(a,b) \sim (a,b)$? That is, do we have $ab = ba$? Yes.

Symmetric: Suppose $(a,b) \sim (c,d)$, then $ad = bc$. Do we have $(c,d) \sim (a,b)$? In other words, is $cb = da$? Yes ✓

Transitive: Suppose $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$, then $ad = bc$ and $cf = ed$, do we have $(a,b) \sim (e,f)$, that is $af = be$?

Yes because $ad = bc \Rightarrow a = \frac{bc}{d}$ (remember $d \neq 0$), so

$$af = \left(\frac{bc}{d}\right) f = \frac{bcf}{d} = \frac{bed}{d} = be \checkmark$$

Hence $\sim$ is an equivalence relation

(b) Notice that $(a,b) \sim (1,2)$ if and only if $a(2) = b(1) \Rightarrow b = 2a$, so examples are $(1,2)$ (which corresponds to $\frac{1}{2}$), $(2,4)$ (which corresponds to $\frac{2}{4} = \frac{1}{2}$), $(3,6)$, $(-1,-2)$, etc.

AP 2:

Date: Friday, September 10, 2021.

(a)

$$\begin{aligned} |x| &= |x - y + y| \\ &\le |x-y| + |y| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

(b)

$$\begin{aligned} |x-t| &= |x-y+y-z+z-t| \\ &\le |x-y| + |y-z| + |z-t| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

(c)

$$\begin{aligned} |y-z| &= |y-x+x-z| \\ &\le |y-x| + |x-z| \\ &< |y-x| + \frac{\epsilon}{2} \\ &< |y-x| + \frac{|y-z|}{2} \end{aligned}$$

Hence $|y-z| \le |y-x| + \frac{1}{2}|y-z| \Rightarrow |y-x| \ge \frac{1}{2}|z-y|$.

AP 3:

(a) We claim that $\sup(A) = 3$.

First of all, for all $n \in \mathbb{N}$, $3 - \frac{2}{n} \le 3$, so 3 is an upper bound for A.

Scratchwork:

$$\begin{aligned} 3 - \frac{2}{n} &> M_1 \\ \Rightarrow -\frac{2}{n} &> (M_1 - 3) \\ \Rightarrow \frac{2}{n} &< (3 - M_1) \\ \Rightarrow \frac{n}{2} &> \frac{1}{3 - M_1} \\ \Rightarrow n &> \frac{2}{3 - M_1} \end{aligned}$$

Now if $M_1 < 3$, then if n is any integer greater than $\frac{2}{3-M_1} > 0$, then $s_1 =: 3 - \frac{2}{n}$ is an element of A such that $s_1 > M_1 \checkmark$

(b) We claim that $\inf(B) = 0$

First of all, for all $x \in \mathbb{R}$, $e^{-x} > 0$, hence 0 is a lower bound for B.

Scratchwork: $e^{-x} < M_1 \Rightarrow -x < \ln(M_1) \Rightarrow x > -\ln(M_1)$

If $M_1 > 0$, then if x is any real number greater than $-\ln(M_1)$, then $s_1 =: e^{-x}$ is an element of B such that $s_1 < M_1 \checkmark$.

(c) I'm claiming that $\sup(C) = \infty$, that is C is not bounded above. In fact for all M, if and n is any **even** integer greater than M, then $n(-1)^n = n > M$, so there exists an element in C that is $\geq M$. where $C = \{n(-1)^n, n \in \mathbb{N}\}$

AP 4: The statement is FALSE. Let $A = B = \{-1, 0\}$, then $\sup(A) = \sup(B) = 0$, But $AB = \{0, 1\}$ and hence $\sup(AB) = 1$

There are other examples, such as $A = \{-1\}$ and $B = \{-1, 1\}$

AP 5

(a) For fixed n and fixed $a_n, \cdots, a_1, a_0$ (not all 0), $a_n x^n + \cdots + a_1 x + a_0$ has at most n roots (by the fundamental theorem of algebra), which is finite, hence countable.

(b) For fixed n, we can write the set of zeros of all polynomials of degree n with integer coefficients as the union over $(a_n, \cdots, a_1) \in \mathbb{Z}^{\star} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ of the set in (a). The set in (a) is countable (finite!) and moreover $\mathbb{Z}^{\star} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ is countable since the product of countable sets is countable. Therefore here we have a countable union of countable sets, which is countable

(c) The set of algebraic numbers is just the union over $n \in \mathbb{N}$ of the set in (b), hence we have a countable union of countable sets, which is countable

AP 6:

Case 1: $i \geq 0$, but then $ii \geq i0$ (by $(O5)$), so $-1 \geq 0 \Rightarrow\Leftarrow$

Case 2: $i \leq 0$, but then $ii \geq i0$, so $-1 \geq 0 \Rightarrow\Leftarrow$.