HOMEWORK 2 – SELECTED BOOK SOLUTIONS

3.8: By contradiction, suppose $a \leq b_1$ for every $b_1 > b$, but $a > b$.

Let $b_1 = \frac{a+b}{2} > b$, then by assumption

$$\begin{aligned} a &\leq b_1 \\ a &\leq \frac{a+b}{2} \\ a &\leq \frac{a}{2} + \frac{b}{2} \\ a - \frac{a}{2} &\leq \frac{b}{2} \\ \frac{a}{2} &\leq \frac{b}{2} \\ a &\leq b \end{aligned}$$

But this contradicts $a > b \Rightarrow\Leftarrow$.

4.14a Let $a \in A$ and $b \in B$ be arbitrary.

Then $a + b \in S = A + B$, so by definition of $\sup(S)$, we have $a + b \leq \sup(S)$, so $a \leq \sup(S) - b$. Since a was arbitrary, the quantity $\sup(S) - b$ is an upper bound for A and so by definition of $\sup(A)$, we have $\sup(A) \leq \sup(S) - b$.

Therefore $b \leq \sup(S) - \sup(A)$. But since b is arbitrary, $\sup(S) - \sup(A)$ is an upper bound for B and therefore by definition of $\sup(B)$,

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we have $\sup(B) \leq \sup(S) - \sup(A)$, so $\sup(A) + \sup(B) \leq \sup(S)$.

On the other hand, if $s \in S$ is arbitrary, then $s = a + b$ where $a \in A$ and $b \in B$, but by definition $a \leq \sup(A)$ and $b \leq \sup(B)$, so $s = a + b \leq \sup(A) + \sup(B)$. But since s was arbitrary, we get $\sup(A)+\sup(B)$ is an upper bound for S, so $\sup(S) \leq \sup(A)+\sup(B)$ by definition of sup.

Therefore, $\sup(A + B) = \sup(S) = \sup(A) + \sup(B)$