Peiyan Li
uni: Pl2801

MATH S4062 – HOMEWORK 2

- **Chapter 7:** 6, 15, 18

Please **also** do the additional problems below.

Additional Problem 1: Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $|f'(x)| < 1$ for all x, but f has no fixed point. (I recommend playing around with exponential functions before looking at the hint)

Additional Problem 2: Show that the sequence f_n defined by

$$f_n(x) = \cos(x+n) + \ln\left(1 + \frac{\sin(nx)}{\sqrt{n+2}}\right)$$

Is equicontinuous on $[0, 2\pi]$

Date: Due: Tuesday, July 12, 2022.

6. Prove that the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{x^2+n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

1) does not converge absolutely for any value of x

$\forall x$, we have the absolute value of each term:

$$\left|\frac{x^2+n}{n^2}\right| \geq \frac{n}{n^2}=\frac{1}{n}$$

By Comparison Test, $\because \sum\frac{1}{n}$ divergent $\therefore \sum\left|\frac{x^2+n}{n^2}\right|$ divergent.

So It does not converge absolutely for any value of x

2) converges uniformly in every bounded interval.

Denoted $a_n=\dfrac{x^2+n}{n^2}=\dfrac{x^2}{n^2}+\dfrac{1}{n}$

then the series could be write as $\sum_{n=1}^{\infty}(-1)^n a_n$

$$S_n=\sum_{i=1}^{n}(-1)^i a_i = -1+x^2\left(\frac{1}{4}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}\cdots\right)+\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}\cdots\right)$$

$$= -1+x^n\sum_{i=2}^{n}(-1)^i\frac{1}{i^2}+\sum_{i=2}^{n}(-1)^i\frac{1}{i}$$

note as $= -1+x^n\cdot S_{b_n}+S_{c_n}$

CLAIM 1: S_{c_n} converges to c

CLAIM 2: S_{b_n} converges to b

$\Rightarrow$ Proved in next Page

Then $S_n=-1+x^nS_{b_n}+S_{c_n} \longrightarrow f=-1+x^nb+c$

In any bounded interval $[a,b]$, $\exists M$ s.t. $|x^n|\leq M$

$\forall \varepsilon>0$ $\exists N_1>0$ s.t. $|S_{c_n}-c|<\frac{\varepsilon}{2}$ for all $n\geq N_1$ By claim 1

$\exists N_2>0$ s.t. $|S_{b_n}-b|<\frac{\varepsilon}{2M}$ for all $n\geq N_2$ By claim 2

take $N=\max\{N_1,N_2\}$

then $|S_n-f|\leq x^n|S_{b_n}-b|+|S_{c_n}-c|\leq M\frac{\varepsilon}{2M}+\frac{\varepsilon}{2}\leq\varepsilon$

for all x and all $n\geq N$

Which mean $\sum_{n=1}^{\infty}(-1)^n\frac{x^2+n}{n^2}$ converges uniformly in every bounded interval, by hight light

Prove of CLAIM 1:

$$S_{c_n} = \sum_{i=2}^{n} (-1)^i \frac{1}{i} \quad \text{converges.}$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} \cdots\cdots$$

let $b_n = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$

$k = 0, 1, 2 \cdots$

$$S_{c_n} = \begin{cases} b_2 + b_4 + \cdots + b_{2k} & \text{when } n = 2k+2 \\ b_2 + b_4 + \cdots + b_{2k-2} + \frac{1}{2k} & \text{when } n = 2k+1 \end{cases}$$

$b_2 + b_4 + \cdots + b_{2k}$

$$= \frac{1}{2\cdot 3} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{2k(2k+1)} \leq \frac{1}{2^2} + \frac{1}{4^2} + \cdots + \frac{1}{(2k)^2}$$

$$= \frac{1}{4}\left(1 + \frac{1}{1^2} + \cdots + \frac{1}{k^2}\right)$$

coverges

So by comparison test S_{c_n} converges when n is even.

Denoted: $S_{c_n} \to C$ when n is even

then we claim $S_{c_n} \to c$

because these two parts converges

because $\lim_{n\to\infty} S_{c_n} \ (n \text{ odd}) = \lim_{n\to\infty} \left((b_2 + b_4 + \cdots + b_{2k-2}) + \left(\frac{1}{2k}\right) \right)$

$$= \lim_{k\to\infty} (b_2 + b_4 + \cdots + b_{2k+2}) + \lim_{k\to\infty} \frac{1}{2k}$$

$$= C + 0 = C$$

Thus S_{c_n} converges, denote as $S_{c_n} \to C$ as $n \to \infty$

Proof of CLAIM 2.

$$S_{b_n} = \sum_{i=2}^{n} (-1)^i \frac{1}{i^2}$$

$\left|(-1)^i \frac{1}{i^2}\right| \leq \frac{1}{i^2}$ while $\sum_{i=1}^{n} \frac{1}{i^2}$ converges. So, S_{b_n} converges

By comparison test, denote as $S_{b_n} \to b$ as $n \to \infty$

15. Suppose f is a real continuous function on R^1, $f_n(t) = f(nt)$ for $n = 1, 2, 3, \ldots$, and $\{f_n\}$ is equicontinuous on $[0, 1]$. What conclusion can you draw about f?

f must be constant in $\mathbb{R}^+$

① f_n equicontinuous $\Rightarrow$ f constant when $f \geq 0$

By assume to contrary, we assume exists $z_1, z_2 \geq 0$, $z_1 \neq z_2$ s.t.

$$f(z_1) = a \neq b = f(z_2),$$

take $\varepsilon = \frac{1}{2}|a-b|$

$\forall \delta > 0 \quad \exists n \in \mathbb{N}$ s.t. $\frac{|z_1 - z_2|}{n} < \delta$ and $\left|\frac{z_1}{n}\right| < 1$, $\left|\frac{z_2}{n}\right| < 1$

take $x = \frac{z_1}{n}$, $y = \frac{z_2}{n}$, we have $|x-y| = \frac{1}{n}|z_1 - z_2| < \delta$

But $|f_n(x) - f_n(y)| = |f(z_1) - f(z_2)| = |a-b| > \frac{1}{2}|a-b| = \varepsilon$

thus $\{f_n\}$ is not equicontinuous, leads to contradiction $\rightarrow\!\!\!\times$

② f constant when $f \geq 0$ $\Rightarrow$ f_n equicontinuous

$\forall \varepsilon > 0 \quad \exists \delta = \frac{1}{2}$

s.t. $\forall |x-y| < \frac{1}{2}$, $x, y \in [0,1]$

$\forall n$, $nx \geq 0$, $ny \geq 0$, $f(nx) = f(ny)$

$|f(nx) - f(ny)| = 0 < \varepsilon$, thus f_n equicontinuous.

In conclusion: f must be constant on $[0, \infty)$ $\times\!\!\!\times$

18. Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann-integrable on $[a, b]$, and put

$$F_n(x) = \int_a^x f_n(t)\,dt \qquad (a \le x \le b).$$

Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$.

We only need to show that $F_n(x)$ is bounded, equicontinuous on $C[a,b]$

1) $F_n(x)$ is bounded

Because f_n is uniformly bounded, say $|f_n| < M$

then $F_n(x) = \int_a^x f_n(t)\,dt \le \int_a^x M\,dt \le \int_a^b M\,dt = M(a-b)$

So $F_n(x)$ is bounded

2) $F_n(x)$ is equicontinuous

Because $\{f_n\}$ be Riemann-integrable on $[a,b]$

By FTC,

$(F_n(x))' = f(x)$

$\forall \varepsilon > 0 \quad \exists \delta = \dfrac{\varepsilon}{M}$

$\forall x, y, \ |x-y| < \delta, \ \forall n$

$|f_n| < M$,

$|F_n(x) - F_n(y)| \le f(z)|x-y| \qquad z \in [x, y]$ by MVT

$\le M\delta$

$\le \varepsilon$

By the hightlight part, $F_n(x)$ is equicontinuous, $F_n(x) \in C[a,b]$

By 1) 2) $F_n(x)$ is bounded, equicontinuous on $C[a,b]$,

By A-A Theorem, there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$.

which ends the proof ※

Additional Problem 1: Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $|f'(x)| < 1$ for all x, but f has no fixed point. (I recommend playing around with exponential functions before looking at the hint)

$|f'(x)| < 1$ But f is not a constraction

(Otherwise if f is constraction, $\mathbb{R}$ is also complete
then by Banach fixed point Therom, f has fixed point)

we need $\sup |f'(x)| = 1$

notice $\frac{e^x}{e^x+1} < 1$ but $\sup \left|\frac{e^x}{e^x+1}\right| = 1$

then we could constract $f(x)$ by

taking $f(x) = \ln(e^x+1)$ then $f'(x) = e^x \cdot \frac{1}{e^x+1} = \frac{e^x}{e^x+1}$

$f(x) = x$

$x = \ln(e^x+1) \Leftrightarrow e^x = e^{\ln(e^x+1)} \Leftrightarrow e^x = e^x+1$

which is impossible

Formally:

$f(x) = \ln(e^x+1)$ is the function we want.

$f'(x) = e^x \frac{1}{e^x+1} = \frac{e^x}{e^x+1} < \frac{e^x+1}{e^x+1} = 1$

and if $f(x)$ has fixed point

$f(x) = x \Leftrightarrow x = \ln(e^x+1) \Leftrightarrow e^x = e^x+1 \Leftrightarrow 1 = 0$ $\times$

So $f(x)$ hasn't got fixed point

Additional Problem 2: Show that the sequence f_n defined by

$$f_n(x) = \cos(x+n) + \ln\left(1 + \frac{\sin(nx)}{\sqrt{n+2}}\right)$$

Is equicontinuous on $[0, 2\pi]$

We only need to show that $f_n(x)$ coverages uniformly in $[0, 2\pi]$

$\forall n \quad |f_n(x)| \leq 1 + \ln\left(1 + \frac{1}{\sqrt{n+2}}\right) \quad \ln(1+x) < x$

$\leq 1 + \frac{1}{\sqrt{n+2}} \longrightarrow 1$ as $n \to \infty$

Then by [Weierstraß M-test], in $[0, 2\pi]$

$|f_n(x)| \leq 1 + \frac{1}{\sqrt{n+2}}$ where $1 + \frac{1}{\sqrt{n+2}} \to 1$ as $n \to \infty$,

$f_n(x)$ coverages uniformly in $[0, 2\pi]$

thus, it is equicontinuous on $[0, 2\pi]$ as well.