HOMEWORK 3 – SELECTED BOOK SOLUTIONS

4.16 Let $A = \{r \in \mathbb{Q} \mid r < a\}$. We claim that a is the least upper bound of A

Upper Bound: First of all, for all $r \in A$, $r < a$ (by definition) so A is bounded above by a

Least upper bound: Suppose $M_1 < a$. Then by denseness of $\mathbb{Q}$ there is a rational r such that $M_1 < r < a$. But since in particular $r < a$, we have $r \in A$, and therefore r is an element of A that is $> M_1$✓

Hence a is the least upper bound of A.

7.4a Let $x_n = \frac{\sqrt{2}}{n}$ which are irrational, because otherwise $nx_n = \sqrt{2}$ would be rational (being the product of two rational numbers), but $\lim_{n\to\infty} x_n = 0$, which is rational

7.4b Let $x_n = \left(1 + \frac{1}{n}\right)^n$, which is rational (being the product of rational numbers), but $\lim_{n\to\infty} x_n = e$, which is irrational.

Note: Another solution would be $x_n =$ expansion of π up to the nth decimal place, like $x_1 = 3.1$, $x_2 = 3.14$, $x_3 = 3.141$ then each x_n is rational but x_n converges to π, which is irrational.

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8.4

Scratchwork:

$$|s_n t_n| = |s_n| \, |t_n| \leq |s_n| \, M < \epsilon \Rightarrow |s_n| < \frac{\epsilon}{M}$$

Proof: Let $\epsilon > 0$ be given. Since $s_n \to 0$ we know that there is an N such that for all n, if $n > N$, then $|s_n| < \frac{\epsilon}{M}$. With that same N, if $n > N$, we therefore get:

$$|s_n t_n| = |s_n| \, |t_n| \leq |s_n| \, M < \left(\frac{\epsilon}{M}\right) M = \epsilon \checkmark$$

8.5b In my opinion, this is easier to prove directly:

Let $\epsilon > 0$ be given, then there is N such that if $n > N$, then $|t_n| = t_n < \epsilon$. With that same N, if $n > N$, then

$$|s_n| \leq t_n < \epsilon \checkmark$$

8.7b Suppose $\lim_{n \to \infty} \sin\left(\frac{\pi n}{3}\right) = a$. Then for all $\epsilon > 0$ there is N such that if $n > N$, then

$$\left| \sin\left(\frac{\pi n}{3}\right) - a \right| < \epsilon$$

But now let n be any integer $> N$ that is a multiple of 6, then

$$\left| \sin\left(\frac{\pi n}{3}\right) - a \right| = |-a| = |a| < \epsilon$$

So $-\epsilon < a < \epsilon$

Similarly, let n be any integer $> N$ that is congruent to 1 modulo 6, then

$$\left|\sin\left(\frac{\pi n}{3}\right) - a\right| = \left|\frac{\sqrt{3}}{2} - a\right| = \left|a - \frac{1}{2}\right| < \epsilon$$

So $-\epsilon < a - \frac{\sqrt{3}}{2} < \epsilon \Rightarrow \frac{\sqrt{3}}{2} - \epsilon < a < \frac{\sqrt{3}}{2} + \epsilon$.

Now choose $\epsilon > 0$ such that $\epsilon \leq \frac{\sqrt{3}}{2} - \epsilon$, that is $\epsilon \leq \frac{\sqrt{3}}{4}$, then we get the contradiction:

$$a < \epsilon \leq \frac{\sqrt{3}}{2} - \epsilon < a \Rightarrow a < a \Rightarrow\Leftarrow$$

Hence $\lim_{n\to\infty} \sin\left(\frac{\pi n}{3}\right)$ doesn't exist. □

8.8a

Scratchwork: Notice that

$$\begin{aligned}
\left|\sqrt{n^2+1} - n - 0\right| &= \left|\sqrt{n^2+1} - n\right| \\
&= \left|\left(\sqrt{n^2+1} - n\right)\left(\frac{\sqrt{n^2+1} + n}{\sqrt{n^2+1} + n}\right)\right| \\
&= \left|\frac{n^2+1-n^2}{\sqrt{n^2+1} + n}\right| \\
&= \left|\frac{1}{\sqrt{n^2+1} + n}\right| \\
&= \frac{1}{\underbrace{\sqrt{n^2+1}}_{\geq 0} + n} \\
&< \frac{1}{n} < \epsilon
\end{aligned}$$

Which implies $n > \frac{1}{\epsilon}$.

Proof: Let $\epsilon > 0$ be given, and let $N = \frac{1}{\epsilon}$. Then if $n > N$, then we get

$$\begin{aligned}\left|\sqrt{n^2+1} - n - 0\right| &= \left|\sqrt{n^2+1} - n\right| \\ &= \frac{1}{\left|\sqrt{n^2+1} + n\right|} \\ &= \frac{1}{\sqrt{n^2+1} + n} \\ &< \frac{1}{n} \\ &= \epsilon \checkmark\end{aligned}$$

8.9.a Let $\epsilon > 0$ be arbitrary, then there is N such that if $n > N$, then $|s_n - s| < \epsilon$, where $s = \lim_n s_n$.

However:

$$|s_n - s| < \epsilon \Rightarrow |s - s_n| < \epsilon \Rightarrow -\epsilon < s - s_n < \epsilon \Rightarrow s > s_n - \epsilon$$

In particular, if you pick n such that $n > N$ and $s_n \geq a$ (which we can since there are only finitely many n such that $s_n < a$), $s > s_n - \epsilon \geq a - \epsilon$, hence $s > a - \epsilon$.

Since ϵ was arbitrary, we ultimately get $s \geq a$. $\square$

Note: If you want to make that last step more elegant, suppose $s < a$ and choose ϵ such that $a - \epsilon > s$, that is $\epsilon < a - s$, then $s > a - \epsilon > s \Rightarrow s > s \Rightarrow \Leftarrow$.

8.10 Let $\epsilon > 0$ be TBA, then since $\lim_{n\to\infty} s_n = s$, there is N such that if $n > N$, then $|s_n - s| < \epsilon$.

But

$$|s_n - s| < \epsilon \Rightarrow -\epsilon < s_n - s < \epsilon \Rightarrow s - \epsilon < s_n < s + \epsilon$$

In particular, we get $s_n > s - \epsilon$

Now choose ϵ such that $s - \epsilon \geq a$, that is $\epsilon \leq s - a$ (Which we can do since $s - a > 0$ by assumption), then for $n > N$, we get

$$s_n > s - \epsilon \geq a$$

So $s_n > a$ for $n > N$ □