pl2801 (uni)

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MATH S4062 – HOMEWORK 3

- **Chapter 7:** 20
- **Chapter 8:** 1, 3

Please **also** do the additional problems below.

Additional Problem 1: Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0 \text{ and } \lim_{n\to\infty} n \int_0^1 x^n f(x)dx = f(1)$$

Additional Problem 2: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function of period 2π such that for all integers $n \geq 0$, we have

$$\int_0^{2\pi} f(x)\sin(nx)dx = 0 \text{ and } \int_0^{2\pi} f(x)\cos(nx)dx = 0$$

Show that f is identically zero (see hints)

Date: Due: Friday, July 15, 2022.

20. If f is continuous on $[0, 1]$ and if

$$\int_0^1 f(x)x^n\,dx = 0 \qquad (n = 0, 1, 2, \ldots),$$

prove that $f(x) = 0$ on $[0, 1]$. *Hint:* The integral of the product of f with any polynomial is zero. Use the Weierstrass theorem to show that $\int_0^1 f^2(x)\,dx = 0$.

By Weierstrass theorem, f is continuous on $[0,1]$, there exists a sequence of polynomial $\{P_n\}$ which uniformly converges to f

Because $\int_0^1 f(x)x^n dx = 0$, for every P_n, $\int_0^1 f(x) P_n(x)\,dx = 0$

And $P_n \to f$ Uniformly, so

$$\begin{aligned}
0 &= \lim_{n\to\infty} 0 && \Big\downarrow \int_0^1 f(x)P_n(x)dx = 0 \\
&= \lim_{n\to\infty}\int_0^1 f(x)P_n(x)\,dx && \Big\downarrow \text{Uniformly converges} \\
&= \int_0^1 f(x) \lim_{n\to\infty} P_n(x)\,dx && \Big\downarrow \text{Weierstrass Theorem} \\
&= \int_0^1 f(x) f(x)\,dx \\
&= \int_0^1 (f(x))^2 dx
\end{aligned}$$

thus $f(x) = 0$ on $[0,1]$

note: otherwise if $\exists\, x_0$ s.t. $f(x_0) = a \neq 0$

Because $f(x)$ is continuous

$\exists\, \delta > 0$ s.t. $\forall x \in B_\delta(x_0)$, we have $f(x) \neq 0$

then $\int_{x_0-\delta}^{x_0+\delta} (f(x))^2 dx$ strictly > 0

notice that $\int_0^{x_0-\delta} (f(x))^2 dx \geq 0$ & $\int_{x+\delta}^1 (f(x))^2 dx \geq 0$

we have $\int_0^1 (f(x))^2 dx > 0$, $\times$.

1. Define

$$f(x)=\begin{cases} e^{-1/x^2} & (x\neq 0),\\ 0 & (x=0).\end{cases}$$

Prove that f has derivatives of all orders at $x=0$, and that $f^{(n)}(0)=0$ for $n=1,2,3,\ldots$.

Prove by induction:

1) Base case: $f(0)=0$ $\quad f'(x)=2x^{-3}e^{-\frac{1}{x^2}} = \frac{P_0(x)}{x^3}e^{-\frac{1}{x^2}}$

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x}=\lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{e^{-\frac{1}{x^2}}}{x}$$

① $\lim_{x\to 0^+}\frac{e^{-\frac{1}{x^2}}}{x}=0$

notice that

$$0\leqslant \frac{1}{x}\leqslant x\cdot\frac{1}{x^2}\cdot\frac{1}{1!}\leqslant x\cdot\sum_{i=0}^{\infty}\frac{1}{i!}\left(\frac{1}{x^2}\right)^i = x\cdot e^{\frac{1}{x^2}}$$

thus $0\leqslant\frac{1}{x}\leqslant x\cdot e^{\frac{1}{x^2}} \Rightarrow 0\leqslant\frac{e^{-\frac{1}{x^2}}}{x}\leqslant x \Rightarrow 0\leqslant\lim_{x\to 0^+}\frac{e^{-\frac{1}{x^2}}}{x}\leqslant\lim_{x\to 0^+}x=0$

$\Rightarrow \lim_{x\to 0^+}\frac{e^{-\frac{1}{x^2}}}{x}=0$

② $\lim_{x\to 0^-}\frac{e^{-\frac{1}{x^2}}}{x}=0$

$$0\geqslant\frac{1}{x}\geqslant x\,\frac{1}{x^2}\,\frac{1}{1!}\geqslant x\sum_{i=1}^{\infty}\frac{1}{i!}\left(\frac{1}{x^2}\right)^i = x\cdot e^{\frac{1}{x^2}}$$

thus $0\geqslant\frac{1}{x}\geqslant x\cdot e^{\frac{1}{x^2}} \Rightarrow 0\geqslant\frac{e^{-\frac{1}{x^2}}}{x}\geqslant x \Rightarrow 0\geqslant\lim_{x\to 0^-}\frac{e^{-\frac{1}{x^2}}}{x}\geqslant\lim_{x\to 0^-}x=0$

$\Rightarrow \lim_{x\to 0^-}\frac{e^{-\frac{1}{x^2}}}{x}=0$

Combine ① and ② we have $\lim_{x\to 0^+}\frac{e^{-\frac{1}{x^2}}}{x}=\lim_{x\to 0^-}\frac{e^{-\frac{1}{x^2}}}{x}=0$

which means $f'(0)$ exists and $f'(0)=0$

2) Induction we assume $f^{(n)}(0)=0$ and $f^{(n)}(x)=\frac{P_{2n-2}(x)}{x^{3n}}e^{-\frac{1}{x^2}}$

considering $f^{(n+1)}$

$$f^{(n+1)}(x) = \frac{-3n\cdot P_{2n-2}(x)}{x^{3n+1}}e^{-\frac{1}{x^2}} - 2x^{-3}e^{-\frac{1}{x^2}}\cdot\frac{P_{2n-2}(x)}{x^{3n}}$$

$$= e^{-\frac{1}{x^2}}\,\frac{-3n\,x^2 P_{2n-2}(x) - 2P_{2n-2}(x)}{x^{3(n+1)}}$$

$$= e^{-\frac{1}{x^2}}\,\frac{P_{2n}(x)}{x^{3(n+1)}} = \frac{P_{2(n+1)-2}(x)}{x^{3(n+1)}}\,e^{-\frac{1}{x^2}}$$

$$f^{(n+1)}(0) = \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x}$$

$$= \lim_{x\to 0} \frac{f^{(n)}(x)}{x} = \lim_{x\to 0} P_{2n-2}(x) \frac{e^{-\frac{1}{x^2}}}{x^{3n+1}} \overset{\text{note as}}{=\!=} \lim_{x\to 0} A \cdot B$$

CLAIM: $\forall m \in N^+$, $\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^m} = 0$

① $\lim_{x\to 0^+} \frac{e^{-\frac{1}{x^2}}}{x^m} = 0$

proof: $\exists n$ s.t. $2n-2 \leq m < 2n$, $2n-m > 0$

$$0 \leq \frac{1}{x^m} \leq n!\, x^{2n-m} \frac{1}{n!} \frac{1}{x^{2n}}$$

$$\leq n!\, x^{2n-m} \sum_{i=1}^{\infty} \frac{1}{n!} \left(\frac{1}{x^2}\right)^n$$

$$= n!\, x^{2n-m} \cdot e^{\frac{1}{x^2}}$$

$$\Rightarrow 0 \leq \frac{e^{-\frac{1}{x^2}}}{x^m} \leq n!\, x^{2n-m} \text{ for } \forall x \Rightarrow \lim_{x\to 0^+} \frac{e^{-\frac{1}{x^2}}}{x^m} = 0$$

② $\lim_{x\to 0^-} \frac{e^{-\frac{1}{x^2}}}{x^m} = 0$

$\exists n$ s.t. $2n-2 \leq m < 2n$, $2n-m > 0$

$$0 \geq \frac{1}{x^m} \geq n!\, x^{2n-m} \frac{1}{n!} \frac{1}{x^{2n}} \geq n!\, x^{2n-m} \cdot e^{\frac{1}{x^2}}$$

$$\Rightarrow 0 \geq \frac{e^{-\frac{1}{x^2}}}{x^m} \geq n!\, x^{2n-m} \text{ for } \forall x \Rightarrow \lim_{x\to 0^-} \frac{e^{-\frac{1}{x^2}}}{x^m} = 0$$

with ① ②, claim is proved

So

$$f^{(n+1)}(0) = \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x}$$

$$= \lim_{x\to 0} \frac{f^{(n)}(x)}{x} = \lim_{x\to 0} P_{2n-2}(x) \frac{e^{-\frac{1}{x^2}}}{x^{3n+1}} \overset{\text{note as}}{=\!=} \lim_{x\to 0} A \cdot B$$

$\lim_{x\to 0} A = 0$

$\lim_{x\to 0} B = \frac{e^{-\frac{1}{x^2}}}{x^{3n+1}} = 0$ by Claim above, thus $f^{(n+1)}(0) = 0$

3) Conclusion: $f^{(n)}(0) = 0$ for all $n \in N^+$

3. **Prove that**

$$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$$

if $a_{ij} \geq 0$ for all i and j (the case $+\infty = +\infty$ may occur).

By Fubini for Series, we know that if $\sum_i \sum_j a_{ij} < \infty$

then $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$

CLAIM: if $\sum_i \sum_j a_{ij} = +\infty$ then $\sum_j \sum_i a_{ij} = +\infty$

Assume to the contrary that $\sum_j \sum_i a_{ij} = b < +\infty$

then by Fubini for Series, we have

$\sum_i \sum_j a_{ij} = b$ as well, contradicts with $\sum_i \sum_j a_{ij} = +\infty$

So the equation is proved no matter $\sum_i \sum_j a_{ij} < \infty$ or $= +\infty$ #

Additional Problem 1: Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0 \text{ and } \lim_{n\to\infty} n \int_0^1 x^n f(x)dx = f(1)$$

$f: [0,1] \to \mathbb{R}$ and f continuous, thus f is bounded.

let $|f(x)| \leq M$ for all x

$$\int_0^1 |x^n f(x)| dx \leq \int_0^1 x^n M dx$$

$$= M \int_0^1 x^n dx$$

$$= M \cdot \frac{1}{n+1} = \frac{M}{n+1}$$

$$0 < \lim_{n\to\infty} \int_0^1 |x^n f(x)| dx \leq \lim_{n\to\infty} \frac{M}{n+1} = 0$$

thus $\lim_{n\to\infty} \int_0^1 x^n f(x) dx = 0$

Additional Problem 1: Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0 \text{ and } \lim_{n\to\infty} n \int_0^1 x^n f(x)dx = f(1)$$

By Approximate Theorem, $\exists \{P_m\}$ Poly s.t. $P_m \to f$ uniformly.

And for each $P_m(x)$, we have following:

$$\lim_{n\to\infty} n \int_0^1 x^n P_m(x) dx$$

$$= \lim_{n\to\infty} n \int_0^1 x^n (a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0) dx$$

$$= \lim_{n\to\infty} n \cdot \left(\frac{a_m}{n+m+1} + \frac{a_{m-1}}{n+m} + \cdots + \frac{a_0}{n+1} \right)$$

$$= a_m \lim_{n\to\infty} \left(\frac{n}{n+m+1}\right) + a_{m-1} \lim_{n\to\infty} \left(\frac{n}{n+m}\right) + \cdots + a_0 \lim_{n\to\infty} \left(\frac{n}{n+1}\right)$$

$$= a_m + a_{m-1} + \cdots + a_0 = P_m(1)$$

Thus: $\lim_{n\to\infty} n \int_0^1 x^n f(x) dx$) Weierstrass

$= \lim_{n\to\infty} n \int_0^1 x^n \lim_{m\to\infty} P_m(x) dx$) $P_m \to f$ Uniform converges

$= \lim_{n\to\infty} \lim_{m\to\infty} n \int_0^1 x^n P_m(x) dx$) $\lim_{m\to\infty} n \int_0^1 x^n P_m(x) dx < \infty$ ①

and $\lim_{n\to\infty} n \int_0^1 x^n P_m(x) dx = P_m(1) < \infty$

$= \lim_{m\to\infty} \lim_{n\to\infty} n \int_0^1 x^n P_m(x) dx$

$= \lim_{m\to\infty} P_m(1)$

$= f(1)$

①: $\lim_{m\to\infty} n \int_0^1 x^n P_m(x) dx = n \int_0^1 x^m f(x) dx < \infty$

Additional Problem 2: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function of period 2π such that for all integers $n \geq 0$, we have

$$\int_0^{2\pi} f(x)\sin(nx)dx = 0 \text{ and } \int_0^{2\pi} f(x)\cos(nx)dx = 0$$

Show that f is identically zero (see hints)

Additional Problem 2: First show it's true for trigonometric polynomials, that is polynomials of the form

$$\sum_{k=0}^{n} a_k\cos(kx) + b_k\sin(kx)$$

(Where $b_0 = 0$). You're allowed to use without proof that

$$\int_0^{2\pi} \cos(mx)\cos(nx)dx = 0 \text{ if } m \neq n$$
$$\int_0^{2\pi} \sin(mx)\sin(nx)dx = 0 \text{ if } m \neq n$$
$$\int_0^{2\pi} \cos(mx)\sin(nx)dx = 0 \text{ for all } m, n$$

Then you can use without proof the fact that trigonometric polynomials are dense in the space of continuous periodic functions of period 2π with the sup norm. (this follows from the Stone WeierstraßTheorem).

For $P(x) = \sum_{k=0}^{m} a_k\cos(kx) + b_k\sin(kx)$

$\int_0^{2\pi} P(x)\sin(nx)dx$ ①

$= \int_0^{2\pi}\left(\sum_{k=0}^{m} a_k\cos(kx) + b_k\sin(kx)\right)\sin(nx)dx$

$= \int_0^{2\pi} b_n\sin^2(nx)\,dx = b_n\int_0^{2\pi}\sin^2(nx)\,dx = 0$

for every n, $\int_0^{2\pi}\sin^2(nx)dx = \pi$ when $n \neq 0$

thus $b_n = 0$ for every $n = 1, 2, \cdots, m$.

Similarly

$\int_0^{2\pi} P(x)\cos(nx)dx$ ②

$= \int_0^{2\pi} a_n\cos^2(nx)dx = a_n\int_0^{2\pi}\cos^2(nx)\,dx = 0$

$\int_0^{2\pi}\cos^2(nx)dx \begin{cases} = \pi & \text{when } n \neq 0 \\ = 2\pi & \text{when } n = 0 \end{cases}$

thus $a_n = 0$ for every $n = 1, 2, \cdots m$

Thus for $P(x) = \sum_{k=0}^{m} a_k\cos(kx) + b_k\sin(kx)$

$\int_0^{2\pi} P(x)\sin(nx)dx = 0$ & $\int_0^{2\pi} P(x)\cos(nx)dx = 0 \Rightarrow P(x) = 0$

By Stone – Weierstrass Theorem,

$\forall f\ \exists\{P_n\}$, every P_i is trigonometric poly, and $P_n \to f$ Uniformly by sup norm.

Denote above P_n as :

$$P_n(x) = \sum_{k=0}^{m_n} a_{n,k}\cos(kx) + b_{n,k}\sin(kx)$$

$\forall k \in \mathbb{N}^+$,

$0 = \int_0^{2\pi} f(x)\sin(kx)\,dx = \int_0^{2\pi}\lim_{n\to\infty} P_n(x)\sin(kx)dx$) Uniformly converges

$= \lim_{n\to\infty}\int_0^{2\pi} P_n(x)\sin(kx)dx = \lim_{n\to\infty} b_{n,k}\pi$ by ①

$\Rightarrow \forall k > 0\ \lim_{n\to\infty} b_{n,k} = 0$

$0 = \int_0^{2\pi} f(x)\cos(kx)dx = \int_0^{2\pi}\lim_{n\to\infty} P_n(x)\sin(kx)\,dx$

$= \lim_{n\to\infty}\int_0^{2\pi} P_n(x)\sin(kx)dx = \lim_{n\to\infty} a_{n,k}\pi$ by ②

$\Rightarrow \forall k > 0\ \lim_{n\to\infty} a_{n,k} = 0$

thus $f(x) = \lim_{n\to\infty} P_n(x) = \sum_{k=0}^{\lim_{n\to\infty} m_n}\lim_{n\to\infty} a_{n,k}\cos(kx) + \lim_{n\to\infty} b_{n,k}\sin(kx)$

$= \sum_{k=0}^{\infty} 0\cdot\cos(kx) + 0\cdot\sin(kx)$

$= 0$. which ends the proof ✱