MATH S4062 – HOMEWORK 3

- **Chapter 7:** 20
- **Chapter 8:** 1, 3

Please **also** do the additional problems below.

Additional Problem 1: Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0 \text{ and } \lim_{n\to\infty} n \int_0^1 x^n f(x)dx = f(1)$$

Additional Problem 2: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function of period 2π such that for all integers $n \geq 0$, we have

$$\int_0^{2\pi} f(x)\sin(nx)dx = 0 \text{ and } \int_0^{2\pi} f(x)\cos(nx)dx = 0$$

Show that f is identically zero (see hints)

Date: Due: Friday, July 15, 2022.

Hints:

Problem 1: Check out this video for a solution: A non-analytic smooth function

Problem 3: This *almost* follows from the theorem discussed in lecture, except when one of the sums is infinite. Optionally, you can also give a direct proof using the monotone sequence theorem.

Additional Problem 1: For the first part, use that f is bounded in order to bound the integral. For the second part, first show it's true for polynomials. That is, if $f(x) = a_0 + a_1 x + \cdots + a_m x^m$ show the integral equals $a_0 + a_1 + \cdots + a_m$

Additional Problem 2: First show it's true for trigonometric polynomials, that is polynomials of the form

$$\sum_{k=0}^{n} a_k \cos(kx) + b_k \sin(kx)$$

(Where $b_0 = 0$). You're allowed to use without proof that

$$\begin{aligned} \int_0^{2\pi} \cos(mx)\cos(nx)dx &= 0 \text{ if } m \neq n \\ \int_0^{2\pi} \sin(mx)\sin(nx)dx &= 0 \text{ if } m \neq n \\ \int_0^{2\pi} \cos(mx)\sin(nx)dx &= 0 \text{ for all } m, n \end{aligned}$$

Then you can use without proof the fact that trigonometric polynomials are dense in the space of continuous periodic functions of period 2π with the sup norm. (this follows from the Stone WeierstraßTheorem).

Optionally, if you wish, you can show that the assumptions of Stone-Weierstraß are satisfied, although it's a bit of a pain (Here the compact set is the circle in $\mathbb{R}^2$ of radius 1, which is in 1-1 correspondence with the interval $[0, 2\pi)$ using the parametrization $p(t) = (\cos(t), \sin(t)))$