MATH 409 – HOMEWORK 4

Readings: Sections 9 and 10. Everything in section 9 is important, and you're responsible for knowing all the proofs in that section. In Section 10, for now just focus on the Monotone Sequence Theorem, and ignore the section on decimal expansions.

- **Section 9:** 9(a)(b), 11(c), 12, 15, AP1, AP2
- **Section 10:** 8, AP3, AP4, AP5, AP6

Note: There are some hints on the last page of the assignment

Additional Problem 1: Show directly (using the definition of a limit) that if $a > 1$, then

$$\lim_{n\to\infty} a^n = \infty$$

Additional Problem 2: Suppose $\lim_{n\to\infty} s_n = \infty$ and (t_n) be a sequence of positive numbers that is bounded above. Show directly (using the definition of a limit) that $\lim_{n\to\infty} \frac{s_n}{t_n} = \infty$

Additional Problem 3: Define (s_n) by $s_1 = 1$ and

$$s_{n+1} = \sqrt{s_n + 1}$$

Show that (s_n) converges and find its limit.

Additional Problem 4: Define (t_n) by $t_1 = 1$ and

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$$t_{n+1} = \left(\frac{n}{n+2}\right) t_n$$

Show that (t_n) converges.

Additional Problem 5: Prove *directly* that if (s_n) is decreasing and bounded below, then (s_n) converges. Do **not** use the fact that $-s_n$ is increasing.

Additional Problem 6: Babylonian square root

In this problem, we will calculate square roots the way that Babylonians did it, without any calculators!

(a) Let $a > 0$ and $b > \sqrt{a}$ be given. Define a sequence (s_n) by $s_1 = b$ and

$$s_{n+1} = \frac{1}{2}\left(s_n + \frac{a}{s_n}\right)$$

Show that (s_n) is a decreasing sequence that converges to $\sqrt{a}$.

(b) The above sequence is an excellent way to approximate square roots. To illustrate, let $a = 2$ and $b = 2$ and calculate s_2, s_3, s_4 (please use approximate values). Compare with the actual answer $\sqrt{2} \approx 1.4142$

Some Hints:

9.12(a): To find N, use the result of problem 10 in section 8. To show $|s_n| < a^{n-N}|s_N|$, write

$$|s_n| = \frac{|s_n|}{|s_{n-1}|}\frac{|s_{n-1}|}{|s_{n-2}|}\cdots\frac{|s_{N+1}|}{|s_N|}|s_N|$$

9.15: Use problem 12.

10.8: Calculate $\sigma_{n+1} - \sigma_n$. At some point, it's useful to write ns_{n+1} as $s_{n+1} + s_{n+1} + \cdots + s_{n+1}$ (n times)

AP1: Write $a^n = (1+b)^n$ for $b = a - 1 > 0$ and use the binomial theorem. This is similar to the proof of Example 8 in lecture, except that here the limit is ∞.

AP3: Show by induction that (s_n) is increasing and $s_n \leq 2$ for all n

AP4: Note $t_n \geq 0$

AP5: Mimic the proof of the Monotone Sequence Theorem (Thm 10.2)

AP6a: First show by induction that $s_n \geq \sqrt{a}$ for all n. In the inductive step consider $s_{n+1} - \sqrt{a}$ and moreover notice that

$$s_n - 2\sqrt{a} + \frac{a}{s_n} = \left(\sqrt{s_n}\right)^2 - 2\left(\sqrt{s_n}\right)\left(\sqrt{\frac{a}{s_n}}\right) + \left(\sqrt{\frac{a}{s_n}}\right)^2$$

Then show that $s_{n+1} \leq s_n$. For this no induction is needed, you can do it directly by calculating $s_{n+1} - s_n$