HOMEWORK 4 – AP SOLUTIONS

AP 1:

STEP 1: Scratch Work

Since $a > 1$, we can write $a = (1 + b)$ for $b = a - 1 > 0$, but then

$$\begin{aligned} a^n =& (1+b)^n \\ =& 1^n + n1^{n-1}b + \text{ Positive Junk} \\ =& 1 + nb + \text{ Positive Junk} \\ >& nb \\ >& M \end{aligned}$$

Which gives $nb > M \Rightarrow n > \frac{M}{b}$, which suggests to use $N = \frac{M}{b}$.

STEP 2: Actual Proof.

Let $M > 0$ be given and let $N = \frac{M}{b}$, then if $n > N$, we have

$$a^n > nb > \left(\frac{M}{b}\right) b = M \checkmark$$

Hence $\lim_{n\to\infty} a^n = \infty$. □

AP 2:

STEP 1: Scratch Work

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Since (t_n) is bounded above, we have $|t_n| = t_n \leq C$ for some $C > 0$ and hence

$$\frac{s_n}{t_n} \geq \frac{s_n}{C} > M$$

Which gives $s_n > CM$.

STEP 2: Actual Proof

Let $M > 0$ be given.

Since (t_n) is bounded above, we know there is $C > 0$ such that for all n, $|t_n| = t_n \leq C$

Now since $s_n \to \infty$, there is N such that if $n > N$, then $s_n > MC$

Now for the same N, if $n > N$, then we have

$$\frac{s_n}{t_n} \geq \frac{s_n}{C} > \frac{MC}{C} = M\checkmark$$

Therefore $\lim_{n\to\infty} \frac{s_n}{t_n} = \infty$ □

AP 3

Claim 1: $s_n \leq 2$ for all n

Proof: Let P_n be the proposition "$s_n \leq 2$"

Base Case: $s_1 = 1 \leq 2\checkmark$

Inductive Step: Suppose P_n is true, that is $s_n \leq 2$, show P_{n+1} is true, that is $s_{n+1} \leq 2$. But then, we get:

$$s_{n+1} = \sqrt{s_n + 1} \leq \sqrt{2+1} = \sqrt{3} \leq \sqrt{4} = 2$$

Hence P_{n+1} is true, so P_n is true for all n, that is $s_n \leq 2$ for all n ✓

Claim 2: (s_n) is increasing

Proof: Let P_n be the proposition $s_{n+1} > s_n$

Base Case: $s_2 = \sqrt{s_1 + 1} = \sqrt{1+1} = \sqrt{2} > 1 = s_1$✓

Inductive Step: Suppose P_n is true, that is $s_{n+1} > s_n$. Show P_{n+1} is true, that is $s_{n+2} > s_{n+1}$. But:

$$\begin{aligned}
s_{n+2} - s_{n+1} &= \sqrt{s_{n+1} + 1} - \sqrt{s_n + 1} \\
&= \left(\sqrt{s_{n+1} + 1} - \sqrt{s_n + 1}\right)\left(\frac{\sqrt{s_{n+1} + 1} + \sqrt{s_n + 1}}{\sqrt{s_{n+1} + 1} + \sqrt{s_n + 1}}\right) \\
&= \frac{\left(\sqrt{s_{n+1} + 1}\right)^2 - \left(\sqrt{s_n + 1}\right)^2}{\sqrt{s_{n+1} + 1} + \sqrt{s_n + 1}} \\
&= \frac{s_{n+1} + 1 - s_n - 1}{\sqrt{s_{n+1} + 1} + \sqrt{s_n + 1}} \\
&= \frac{s_{n+1} - s_n}{\sqrt{s_{n+1} + 1} + \sqrt{s_n + 1}}
\end{aligned}$$

But by the inductive hypothesis, $s_{n+1} - s_n > 0$, and the denominator is positive as well, and so $s_{n+2} - s_{n+1} > 0$, that is $s_{n+2} > s_{n+1}$.

Therefore P_{n+1} is true, and hence P_n is true for all n, that is $s_{n+1} > s_n$ for all n ✓

Therefore since (s_n) is increasing and bounded above (by 2), by the Monotone Sequence Theorem, (s_n) converges to s.

Passing to the limit in the identity $s_{n+1} = \sqrt{s_n + 1}$ we get:

$$\begin{aligned} s =& \sqrt{s+1} \\ s^2 =& s + 1 \\ s^2 - s - 1 =& 0 \\ s =& \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

But $\frac{1-\sqrt{5}}{2} < 0$, but $s_n \geq 0$ for all n and therefore $\boxed{s = \frac{1+\sqrt{5}}{2} = \phi}$ (the golden ratio).

AP 4

First of all, $t_n > 0$ for all n (easy induction). Moreover, for all n,

$$\frac{t_{n+1}}{t_n} = \frac{\left(\frac{n}{n+2}\right) t_n}{t_n} = \frac{n}{n+2} < 1$$

And therefore $t_{n+1} < t_n$, so (t_n) is decreasing.

Since (t_n) is decreasing and bounded below, by the Monotone Sequence Theorem, (t_n) converges.

AP 5

Consider $S = \{s_n \mid n \in \mathbb{N}\}$.

Since (s_n) is bounded below, for some C, we have $s_n \geq C$ for all n, and therefore S is bounded below and therefore has a least upper bound $s = \infty(S)$

Claim: (s_n) converges to s.

Let $\epsilon > 0$ be given, then notice $s + \epsilon > m$ and therefore there is some s_N such that $s_N < s + \epsilon$. But then, since (s_n) is decreasing, if $n > N$, $s_n < s_N < s + \epsilon$, so $s_n < s + \epsilon$, so $s_n - s < \epsilon$. Moreover, since (s_n) is bounded below by s, we also have $s_n \geq s > s - \epsilon$, so $s_n - s > -\epsilon$

Therefore, if $n > N$, $-\epsilon < s_n - s < \epsilon$, so $|s_n - s| < \epsilon$.

Therefore (s_n) converges to m □

AP 6(a)

Claim 1: $s_n \geq \sqrt{a}$ for all n

First of all, $s_1 = b \geq \sqrt{a}$. Moreover, if $m > 1$, then $m = n+1$ for some $n \in \mathbb{N}$, and so:

Inductive Step: Suppose P_n is true, that is $s_n \geq \sqrt{a}$. Show P_{n+1} is true, that is $s_{n+1} \geq \sqrt{a}$, but then:

$$
\begin{aligned}
s_m - \sqrt{a} =& s_{n+1} - \sqrt{a} \\
=& \frac{1}{2}\left(s_n + \frac{a}{s_n}\right) - \sqrt{a} \\
=& \frac{1}{2}\left(s_n - 2\sqrt{a} + \frac{a}{s_n}\right) \\
=& \frac{1}{2}\left((\sqrt{s_n})^2 - 2\sqrt{s_n}\left(\sqrt{\frac{a}{s_n}}\right) + \left(\sqrt{\frac{a}{s_n}}\right)^2\right) \\
=& \frac{1}{2}\left(\sqrt{s_n} - \sqrt{\frac{a}{s_n}}\right)^2 \\
\geq& 0
\end{aligned}
$$

This combined with $s_1 \geq \sqrt{a}$ allows us to conclude that for all $n \in \mathbb{N}$, we have $s_n \geq \sqrt{a}$.

Claim 2: $s_{n+1} \leq s_n$ for all n

But

$$
\begin{aligned}
s_{n+1} - s_n =& \frac{1}{2}\left(s_n + \frac{a}{s_n}\right) - s_n \\
=& \frac{1}{2}\left(-s_n + \frac{a}{s_n}\right) \\
=& \frac{1}{2}\left(\frac{-(s_n)^2 + a}{s_n}\right)
\end{aligned}
$$

But from before $s_n \geq \sqrt{a}$, so $(s_n)^2 \geq a$ and so $-(s_n)^2 + a \leq 0$

And therefore we ge $s_{n+1} - s_n \leq 0$, so $s_{n+1} \leq s_n$

Therefore (s_n) is a nonincreasing sequence that is bounded below by $\sqrt{a}$, and hence (s_n) converges to s.

And passing to the limit in

$$s_{n+1} = \frac{1}{2}\left(s_n + \frac{a}{s_n}\right)$$

We get:

$$\begin{aligned} s =& \frac{1}{2}\left(s + \frac{a}{s}\right) \\ 2s =& \frac{s^2 + a}{s} \\ 2s^2 & s^2 + a \\ s^2 =& a \\ s =& \pm\sqrt{a} \end{aligned}$$

But since $s_n \geq 0$, we ultimately get $s = \sqrt{a}$.

AP 8(B)

$$\begin{aligned} s_1 =& 2 \\ s_2 =& \frac{1}{2}\left(2 + \frac{2}{2}\right) = \frac{3}{2} = 1.5 \\ s_3 =& \frac{1}{2}\left(1.5 + \frac{2}{1.5}\right) \approx 1.41666\ldots \\ s_4 =& \frac{1}{2}\left(1.41666 + \frac{2}{1.41666}\right) \approx 1.4142\ldots \end{aligned}$$