HOMEWORK 4 – SELECTED BOOK SOLUTIONS

9.12(a)

Since $L < 1$, let a be such that $L < a < 1$. Then, by the result of problem 10 in section 8 applied to the sequence $\left|\frac{s_{n+1}}{s_n}\right|$ (which converges to L), we obtain that there is N such that if $n \geq N$, then $\left|\frac{s_{n+1}}{s_n}\right| < a$, that is $|s_{n+1}| < a\,|s_n|$

Now for $n > N$, notice that

$$|s_n| = \frac{|s_n|}{|s_{n-1}|}\frac{|s_{n-1}|}{|s_{n-2}|}\cdots\frac{|s_N+1|}{|s_N|}\,|s_N| < \underbrace{aa\ldots a}_{n-N \text{ times}}\,|s_N| = a^{n-N}\,|s_N|$$

Therefore we get $0 \leq |s_n| < a^{n-N}\,|s_N|$, for $n \geq N$. Now, on the one hand, $\lim_{n\to\infty} 0 = 0$ and,on the other hand, since $a < 1$ and N is fixed, $\lim_{n\to\infty} a^{n-N}\,|s_N| = 0$, and so, by the Squeeze Theorem, $\lim_{n\to\infty} |s_n| = 0$ and so $\lim_{n\to\infty} s_n = 0$. □

10.8

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\begin{aligned}
\sigma_{n+1} - \sigma_n =& \frac{1}{n+1}\left(s_1 + \cdots + s_{n+1}\right) - \frac{1}{n}\left(s_1 + \cdots + s_n\right) \\
=& \frac{n\left(s_1 + \cdots + s_{n+1}\right) - (n+1)\left(s_1 + \cdots + s_n\right)}{n(n+1)} \\
=& \frac{ns_1 + \cdots + ns_n + ns_{n+1} - ns_1 - s_1 - \cdots - ns_n - s_n}{n(n+1)} \\
=& \frac{ns_{n+1} - s_1 - s_2 - \cdots - s_n}{n(n+1)} \\
=& \frac{(s_{n+1} - s_1) + (s_{n+1} - s_2) + \cdots + (s_{n+1} - s_n)}{n(n+1)} \\
\geq& 0
\end{aligned}
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In the last step, we used that each term in the numerator is nonnegative, since (s_n) is nondecreasing. ✓