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Additional Problem 1: Show using geometric sums that if

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} \text{ then } D_N(x) = \frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}$$

Additional Problem 1: Notice $e^{inx} = \left(e^{ix}\right)^n = \omega^n$ where $\omega = e^{ix}$, and split up the sum as:

$$\sum_{n=-N}^{-1} e^{inx} + \sum_{n=0}^{N} e^{inx}$$

Once you add up the two sums, multiply top and bottom by $\omega^{-\frac{1}{2}}$

$e^{inx} = (e^{ix})^n = w^n$ where $w = e^{ix}$

$$D_N(x) = \sum_{n=-N}^{-1} e^{inx} + \sum_{n=0}^{N} e^{inx} = \sum_{n=-N}^{-1} w^n + \sum_{n=0}^{N} w^n$$

Denoted as $A + B$

For A:

$$Aw - A = \sum_{n=-(N-1)}^{0} w^n - \sum_{n=-N}^{-1} w^n$$

$$= w^0 - w^{-N}$$

So $A = \dfrac{1-w^{-N}}{w-1}$

For B

$$B - wB = \sum_{n=0}^{N} w^n - \sum_{n=1}^{N+1} w^{n+1} = 1 - w^{N+1}$$

$$B = \frac{1-w^{N+1}}{1-w} = \frac{w^{N+1}-1}{w-1}$$

Thus $A+B = \dfrac{1-w^{-N}}{w-1} + \dfrac{w^{N+1}-1}{w-1} = \dfrac{w^{N+1}-w^{-N}}{w-1}$

$$= \frac{(w^{N+1}-w^{-N})\,w^{-\frac{1}{2}}}{(w-1)\,w^{-\frac{1}{2}}} = \frac{w^{N+\frac{1}{2}} - w^{-(N+\frac{1}{2})}}{w^{\frac{1}{2}} - w^{-\frac{1}{2}}}$$

$$= \frac{e^{i(N+\frac{1}{2})x} - e^{i(-N-\frac{1}{2})x}}{e^{i\frac{1}{2}x} - e^{i-\frac{1}{2}x}} \ \underset{\text{as}}{\overset{\text{denoted}}{=}} \ \frac{C}{D}$$

By $\cos(-x) = \cos x$, $\sin(-x) = -\sin x$, we have:

$e^{iz} = \cos z + i\sin z$; $e^{i(-z)} = \cos(-z) + i\sin(-z) = \cos z - i\sin(z)$

$e^{iz} - e^{i-z} = 2i\sin z$. Apply this formula to C and D. we have:

So $C = 2i\sin((N+\frac{1}{2})x)$; $D = 2i\sin(\frac{1}{2}x)$

Thus

$$D_N(x) = A+B = \frac{C}{D} = \frac{2i\sin\left(\left(N+\frac{1}{2}\right)x\right)}{2i\sin\left(\frac{1}{2}x\right)} = \frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\left(\frac{1}{2}x\right)}$$

which ends the proof ✗

Additional Problem 2: The **Féjer Kernel** is defined as

$$F_N(x) = \frac{1}{N}\sum_{M=0}^{N-1} D_M(x)$$

Show that $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)dx = 1$ and

$$F_N(x) = \frac{1}{N}\left(\frac{\sin^2\left(\frac{Nx}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}\right)$$

$$\int_{-\pi}^{\pi} F_N(x) = \frac{1}{N}\int \sum_{M=0}^{N-1} D_M(x) = \frac{1}{N}\sum_{M=0}^{N-1}\int_{-\pi}^{\pi} D_M(x)\,dx$$

→ we can do this because $D_M(x)$ is integrable.

$$= \frac{1}{N}\sum_{M=0}^{N-1}\int_{-\pi}^{\pi}\sum_{n=-M}^{M} e^{inx}\,dx$$

$$= \frac{1}{N}\sum_{M=0}^{N-1}\sum_{n=-M}^{M}\int_{-\pi}^{\pi} e^{inx}\,dx$$

we know that
$$\int_{-\pi}^{\pi} e^{inx} = \begin{cases} 0 & \text{when } n \neq 0 \\ 2\pi & \text{when } n = 0 \end{cases}$$

So $$\int_{-\pi}^{\pi} F_N(x) = \frac{1}{N}\sum_{M=0}^{N-1}(2\pi + 0 + 0 + \cdots + 0)$$
$$= \frac{1}{N}\cdot N \cdot 2\pi = 2\pi$$

Thus $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)\,dx = 1$ is proved

From the Previous Question, we know that $D_N(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}$

$$D_M(x) = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{x}{2})}$$

$$F_N(x) = \frac{1}{N}\sum_{M=0}^{N-1} D_M(x) = \frac{1}{N}\sum_{M=0}^{N-1}\frac{\sin(M+\frac{1}{2})x}{\sin\frac{x}{2}}$$

$$= \frac{1}{N}\,\frac{\sum_{M=0}^{N-1}\sin(M+\frac{1}{2})x}{\sin\frac{x}{2}} = \frac{1}{N}\,\frac{\sum_{M=0}^{N-1}\sin(\frac{x}{2})\sin((M+\frac{1}{2})x)}{\sin^2\frac{x}{2}}$$

we have:
$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$,

$$F_N(x) = \frac{1}{N}\cdot\frac{1}{\sin^2\frac{x}{2}}\cdot\sum_{M=0}^{N-1}\sin(\tfrac{x}{2})\sin((M+\tfrac{1}{2})x)$$

$$= \frac{1}{N}\cdot\frac{1}{\sin^2\frac{x}{2}}\cdot\sum_{M=0}^{N-1}\frac{1}{2}(\cos Mx - \cos(M+1)x)$$

$$= \frac{1}{N}\cdot\frac{1}{\sin^2\frac{x}{2}}\cdot\frac{1}{2}(\cos 0 - \cos Nx)$$

$\sin\frac{N}{2}x\sin\frac{N}{2}x = \frac{1}{2}(\cos(0) - \cos(Nx))$

$$= \frac{1}{N}\,\frac{1}{\sin^2\frac{x}{2}}\cdot\frac{1}{2}\left(\cos\left(\tfrac{N}{2}x - \tfrac{N}{2}x\right) - \cos\left(\tfrac{N}{2}x + \tfrac{N}{2}x\right)\right)$$

$$= \frac{1}{N}\left(\frac{\sin^2(\frac{N}{2}x)}{\sin^2\frac{x}{2}}\right)$$

, which ends the proof.

Additional Problem 3: Suppose f is a 2π periodic function that is of class[1] C^k for some $k \geq 1$. Show that there is a constant C (depending on k) such that

$$\left|\hat{f}(n)\right| \leq \frac{C}{|n|^k}$$

[1] This means that $f, f', f'', \cdots f^{(k)}$ exist and are continuous

And deduce from a theorem in lecture that if $k \geq 2$, then the Fourier series of f converges to f uniformly.

CLAIM: $2\pi \hat{f}(n) = \frac{1}{(in)^k} \int_0^{2\pi} f^{(k)}(x) e^{-inx} dx$ for all $k \in N^+$

Prove by induction:

1) Base case

$$2\pi \hat{f}(n) = \int_0^{2\pi} f(x) e^{-inx} dx$$

$$\overset{IBP}{=} \left[f(x) \frac{-e^{inx}}{in} \right]_0^{2\pi} - \int_0^{2\pi} f'(x) \frac{e^{-inx}}{-in} dx = \frac{1}{in} \int_0^{2\pi} f'(x) e^{-inx} dx$$

2) Induction: We assume that claim is true for all $1, 2, \cdots, k-1$,

$$2\pi \hat{f}(n) = \frac{1}{(in)^{k-1}} \int_0^{2\pi} f^{(k-1)}(n) e^{-inx} dx$$

$$\overset{IBP}{=} \frac{1}{(in)^{k-1}} \left[\left[f^{(k-1)}(x) \frac{-e^{inx}}{in} \right]_0^{2\pi} - \int_0^{2\pi} f^{(k)}(x) \frac{e^{-inx}}{-in} dx \right]$$

$$= \frac{1}{(in)^k} \int_0^{2\pi} f^{(k)}(x) e^{-inx} dx$$

3) conclusion: CLAIM is true for all $k \in N^+$

Because $f^{(k)}$ is continuous, $\max f^{(k)}$ in $[0, 2\pi]$ exists, let $C = \max |f^{(k)}(x)|$

Then $2\pi \hat{f}(n) = \frac{1}{(in)^k} \int_0^{2\pi} f^{(k)}(x) e^{-inx} dx$

$$\leq \frac{1}{|n|^k} \int_0^{2\pi} C \left|e^{-inx}\right| dx = \frac{2\pi}{|n|^k} \cdot C$$

$\Rightarrow \hat{f}(n) \leq \frac{C}{|n|^k}$ is proved #

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if $k \geq 2$ $\sum_{n=-\infty}^{+\infty} |\hat{f}(n)| \leq \left(\sum_{n=-\infty}^{+\infty} \frac{1}{n^k} \right) \cdot C$, the right hand side converges.

thus by Weierstrass-M test, $\sum_{n=-\infty}^{+\infty} |\hat{f}(n)| < \infty$,

Thus by Therom in lecture (Uniform convergence), Fourier Series of f converges uniformly to f #

Additional Problem 4: Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of Riemann integrable functions on $[-\pi,\pi]$ such that

$$\lim_{k\to\infty}\int_{-\pi}^{\pi}|f_k(x)-f(x)|\,dx=0$$

Show that for all n, $\lim_{k\to\infty}\hat{f}_k(n)=\hat{f}(n)$ uniformly in n

$$\hat{f}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-inx}dx$$

$$\hat{f}_k(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f_k(x)e^{-inx}dx$$

Because $\lim_{k\to\infty}\int_{-\pi}^{\pi}|f_k(x)-f(x)|dx=0$

Take any $\varepsilon>0$, exists $N\in\mathbb{N}$, such that for all $k>N$, we have

$$\left|\int_{-\pi}^{\pi}|f_k(x)-f(x)|dx-0\right|=\int_{-\pi}^{\pi}|f_k(x)-f(x)|dx<2\pi\varepsilon$$

Then For all n, N is independent of n

$$|\hat{f}_k(n)-\hat{f}(n)|=\frac{1}{2\pi}\int_{-\pi}^{\pi}(f_k(x)-f(x))\,e^{-inx}dx$$

$$\leq\frac{1}{2\pi}\int_{-\pi}^{\pi}|f_k(x)-f(x)|\,|e^{-inx}|\,dx \quad |e^{-inx}|=1$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f_k(x)-f(x)|dx$$

$$<\frac{1}{2\pi}\cdot2\pi\varepsilon=\varepsilon$$

Which means $\lim_{k\to\infty}\hat{f}_k(n)=\hat{f}(n)$ uniformly,

which ends the proof

Additional Problem 5: Let $\{K_n\}_{n=1}^{\infty}$ be a family of functions called **good kernels** with the following properties:

(1) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)dx = 1$ ✓

(2) There is $M > 0$ such that for all n,

$$\int_{-\pi}^{\pi} |K_n(x)|\, dx \leq M$$ ✓

(3) For every $\delta > 0$,

$$\lim_{n\to\infty} \int_{\delta\leq|x|\leq\pi} |K_n(x)|\, dx = 0$$ ✓

Suppose f is a 2π periodic function that is continuous at x, show that

$$\lim_{n\to\infty} (f \star K_n)(x) = f(x)$$

Aside: The Dirichlet kernel is not a good kernel, so we can't apply this result to $S_N(f)$

By definition

$$(f * K_n)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, K_n(x-y)\,dy$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\, K_n(y)\,dy$$

$$(f*K_n)(x) - f(x)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\,K_n(y)\,dy - f(x)\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)\,dy$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x-y) - f(x)\right) K_n(y)\,dy$$

$$|(f*K_n)(x) - f(x)|$$
$$= \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x-y) - f(x))K_n(y)\,dy \right|$$

Because $f(x)$ is continuous at x,

take any $\varepsilon > 0$ exists $\delta > 0$ such that for all y, $|y| < \delta$, we have

$$|f(x-y) - f(x)| < \frac{\varepsilon}{2M}\cdot 2\pi = \frac{\pi\varepsilon}{M}$$

Then for same ε and δ,

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x-y) - f(x))K_n(y)\,dy\right|$$
$$= \left|\frac{1}{2\pi}\left(\int_{|y|<\delta} + \int_{\delta\leq|y|\leq\pi}\right)(f(x-y) - f(x))\,K_n(y)\,dy\right| := |\text{PART A} + \text{PART B}|$$
$$\leq |\text{PART A}| + |\text{PART B}|.$$

PART A: using $f(x)$ is continuous at x

$$\left|\frac{1}{2\pi}\int_{|y|<\delta} (f(x-y) - f(x))\, K_n(y)\,dy\right| \leq \frac{1}{2\pi}\int_{|y|<\delta} |f(x-y) - f(x)|\, K_n(y)\,dy$$
$$< \frac{1}{2\pi}\int_{|y|<\delta} \frac{\pi\varepsilon}{M} K_n(y)\,dy$$
$$= \frac{\varepsilon}{2M}\int_{|y|<\delta} K_n(y)\,dy$$
$$\leq \frac{\varepsilon}{2M}\int_{-\pi}^{\pi} K_n(y)\,dy \leq \frac{\varepsilon}{2M} M = \frac{\varepsilon}{2}$$

PARTB: let $M_f = \sup_x (|f(x)|)$ because $f(x)$ intergrable otherwise convolution cannot be defined.

Because For every $\delta > 0$, $\lim_{n\to\infty} \int_{\delta \le |x| \le \pi} |k_n(x)| dx = 0$ Given. "Good" condition

for above δ and ε

exists $N \in \mathbb{N}$, such that: for every $n \geq N$, we have: $\int_{\delta \le |y| \le \pi} |k_n(y)| dy < \frac{\varepsilon\pi}{2M_f}$

Then,
$$\left| \frac{1}{2\pi} \int_{\pi \geq |y| \geq \delta} (f(x-y) - f(x)) k_n(y) dy \right|$$
$$\leq \frac{1}{2\pi} \int_{\pi \geq |y| \geq \delta} 2M_f |k_n(y)| dy$$

$f(x-y) \le M_f$, $f(x) \le M_f$

$$\leq \frac{M_f}{\pi} \int_{\delta \le |y| \le \pi} |k_n(y)| dy < \frac{\cancel{M_f}}{\cancel{\pi}} \frac{\varepsilon \cancel{\pi}}{2\cancel{M_f}} = \frac{\varepsilon}{2}$$

In Summary:

Take any $\varepsilon > 0$, exists $\delta > 0$ and $N \in \mathbb{N}$ such that for every $n \geq N$

we have
$$|(f * k_n)(x) - f(x)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-y) - f(x)) k_n(y) dy \right|$$
$$= |PARTA + PARTB|$$
$$\leq |PARTA| + |PARTB|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which means

$\lim_{n\to\infty} (f * k_n)(x) = f(x)$. ends the proof

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