MATH S4062 – HOMEWORK 4

Additional Problem 1: Show using geometric sums that if

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} \text{ then } D_N(x) = \frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}$$

Additional Problem 2: The **Féjer Kernel** is defined as

$$F_N(x) = \frac{1}{N}\sum_{M=0}^{N-1} D_M(x)$$

Show that $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)dx = 1$ and

$$F_N(x) = \frac{1}{N}\left(\frac{\sin^2\left(\frac{Nx}{2}\right)}{\sin^2\left(\frac{x}{2}\right)}\right)$$

Additional Problem 3: Suppose f is a 2π periodic function that is of class[1] C^k for some $k \geq 1$. Show that there is a constant C (depending on k) such that

$$\left|\hat{f}(n)\right| \leq \frac{C}{|n|^k}$$

And deduce from a theorem in lecture that if $k \geq 2$, then the Fourier series of f converges to f uniformly.

Date: Due: Tuesday, July 19, 2022.

[1]This means that $f, f', f'', \cdots f^{(k)}$ exist and are continuous

Additional Problem 4: Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of Riemann integrable functions on $[-\pi, \pi]$ such that

$$\lim_{k\to\infty} \int_{-\pi}^{\pi} |f_k(x) - f(x)|\, dx = 0$$

Show that for all n, $\lim_{k\to\infty} \hat{f}_k(n) = \hat{f}(n)$ uniformly in n

Additional Problem 5: Let $\{K_n\}_{n=1}^{\infty}$ be a family of functions called **good kernels** with the following properties:

(1) $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x) dx = 1$

(2) There is $M > 0$ such that for all n,

$$\int_{-\pi}^{\pi} |K_n(x)|\, dx \leq M$$

(3) For every $\delta > 0$,

$$\lim_{n\to\infty} \int_{\delta \leq |x| \leq \pi} |K_n(x)|\, dx = 0$$

Suppose f is a 2π periodic function that is continuous at x, show that

$$\lim_{n\to\infty} \left(f \star K_n\right)(x) = f(x)$$

Aside: The Dirichlet kernel is not a good kernel, so we can't apply this result to $S_N(f)$

Hints:

Additional Problem 1: Notice $e^{inx} = \left(e^{ix}\right)^n = \omega^n$ where $\omega = e^{ix}$, and split up the sum as:

$$\sum_{n=-N}^{-1} e^{inx} + \sum_{n=0}^{N} e^{inx}$$

Once you add up the two sums, multiply top and bottom by $\omega^{-\frac{1}{2}}$

Additional Problem 2: One way to do this is to use the explicit formula for D_N and

$$2\sin(A)\sin(B) = \cos(A-B) - \cos(A+B)$$

Additional Problem 3: Mimic the proof I gave in lecture with f''

Additional Problem 5: Use that $f(x) = \frac{1}{2\pi} f(x) \int_{-\pi}^{\pi} K_n(y)dy$ Then split up the integral of the difference in two parts, one on $|y| < \delta$ and the other one on $\delta \le |y| \le \pi$.