MATH 409 – HOMEWORK 5

Note: This assignment is shorter than usual, to give you some time to breathe after the midterm ☺

Readings: Sections 10 and 11. This time in section 10 focus on Cauchy sequences. Section 11 in the book is a bit awkward; as long as you understand what I did in lecture, you're good. You don't need to read section 12 to do the problem below.

- **Section 10:** AP1, AP2
- **Section 11:** 8, 9, 11, AP3, AP4
- **Section 12:** 4

Additional Problem 1: Show that $\mathbb{Z}$ is complete, that is any Cauchy sequence in $\mathbb{Z}$ converges.

Additional Problem 2: Here's another construction of $\mathbb{Q}$ that doesn't use any cuts or decimal expansions!

Definition: We write $(p_n) \sim (q_n)$ if and only if $\lim_{n\to\infty} p_n - q_n = 0$

Show that $\sim$ is an equivalence relation on the set of all sequences in $\mathbb{Q}$.

Date: Due: Friday, October 8, 2021.

Note: We then define $\mathbb{R}$ as the set of all equivalence classes of Cauchy sequences in $\mathbb{Q}$ (with the equivalence relation above). For example $\pi = [(3, 3.1, 3.14, 3.141, \dots)]$. In other words, a real number *is* just a sequence of rational numbers approximating it! It's like the problem itself *is* the solution!

Additional Problem 3: Suppose (s_n) is a sequence with the property that every subsequence has a further subsequence that converges to s. Prove that (s_n) converges to s.

Additional Problem 4:

(a) Suppose (s_n) is a bounded sequence with the property that every *convergent* subsequence of (s_n) converges to s. Show that (s_n) converges to s.

(b) Show that (a) is false if (s_n) is not bounded

Hints:

AP1: Could two *different* integers ever be close to each other? If you're not convinced, try the definition of a Cauchy sequence, but with $\epsilon = 1$.

AP3: Do it by contradiction: Suppose (s_n) does not converge to s and use this to find a subsequence that has no convergent further subsequence.

AP4: Do it by contradiction: Suppose (s_n) doesn't converge to s and construct a subsequence that doesn't converge to s (it's actually the same subsequence as in AP3). But doesn't the Bolzano-Weierstraß Theorem not apply to that subsequence?