HOMEWORK 5 – AP SOLUTIONS

AP 1

Let (s_n) be a Cauchy sequence in $\mathbb{Z}$. We want to show that (s_n) converges.

But letting $\epsilon = 1$ in the definition of Cauchy, there is N such that if $m, n > N$, then $|s_n - s_m| < 1$.

Letting $m = N + 1$, we get that for all $n > N$, then $|s_n - s_{N+1}| < 1$ which implies that $s_n = s_{N+1}$ since no different integers are less than 1 apart. so (s_n) is just the sequence $(s_1, s_2, \cdots, s_N, s_{N+1}, s_{N+1}, \cdots)$, which is eventually constant and therefore converges to s_{N+1}.

AP 2

Reflexivity: Is $(p_n) \sim (p_n)$? Yes because $p_n - p_n = 0$, and therefore

$$\lim_{n\to\infty} p_n - p_n = \lim_{n\to\infty} 0 = 0$$

Symmetry Suppose $(p_n) \sim (q_n)$, that is $\lim_{n\to\infty} |p_n - q_n| = 0$, but then

$$\lim_{n\to\infty} q_n - p_n = - \lim_{n\to\infty} |p_n - q_n| = -0 = 0$$

Hence $(q_n) \sim (p_n)$ ✓

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Transitivity Suppose $(p_n) \sim (q_n)$ and $(q_n) \sim (r_n)$. But then

$$\begin{aligned}\lim_{n\to\infty} p_n - r_n &= \lim_{n\to\infty} p_n - q_n + q_n - r_n \\ &= \lim_{n\to\infty} p_n - q_n + \lim_{n\to\infty} q_n - r_n \\ &= 0 + 0 \\ &= 0\end{aligned}$$

Hence $(p_n) \sim (r_n)$ ✓

AP 3

Suppose (s_n) doesn't converge to s. Then there is $\epsilon > 0$ such that for all N there is $n > N$ with $|s_n - s| > \epsilon$

In particular, for $N = 1$ there is $n_1 > 1$ with $|s_{n_1} - s| > \epsilon$

And for $N = n_1$ there is $n_2 > n_1$ with $|s_{n_2} - s| > \epsilon$

And, in general, for $N = n_k$ there is $n_{k+1} > n_k$ such that $\left|s_{n_{k+1}} - s\right| > \epsilon$.

Therefore we have inductively constructed a subsequence (s_{n_k}) of (s_n) with $|s_{n_k} - s| > \epsilon$ for all k. But no subsequence of (s_{n_k}) converges to s (since all the terms are at least ϵ away from s) $\square$

AP 4(a)

Suppose s_n doesn't converge to s. Then there is $\epsilon > 0$ such that for all N there is $n > N$ with $|s_n - s| > \epsilon$. Construct the same subsequence (s_{n_k}) as in the previous problem.

Since (s_{n_k}) is bounded, by the Bolzano-Weierstraß Theorem, (s_{n_k}) has a convergent subsequence. By assumption, that (sub-)subsequence must converge to s, but this contradicts $|s_{n_k} - s| > \epsilon$ for all k (and in particular for the sub-subsequence) $\Rightarrow\Leftarrow$ $\square$

AP 4(b)

Consider:

$$s_n = \begin{cases} n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Then every *convergent* subsequence of (s_n) must converge to 0 (because it must eventually end with all 0's), but (s_n) doesn't converge to 0.