HOMEWORK 5 – SELECTED BOOK SOLUTIONS

11.8

First of all, for all sets S, we know that $\inf(S) = -\sup(-S)$. Now if N is arbitrary, this implies in particular that

$$\inf\ \{s_n \mid n > N\} = -\sup\left(-\{s_n \mid n > N\}\right) = -\sup\{-s_n \mid n > N\}$$

Now taking the limit as $N \to \infty$ of both sides, we get:

$$\begin{aligned}
\liminf_{n\to\infty} s_n &= \lim_{N\to\infty} \ \inf\ \{s_n \mid n > N\} \\
&= \lim_{N\to\infty} -\sup\{-s_n \mid n > N\} \\
&= -\lim_{N\to\infty} \sup\{-s_n \mid n > N\} \\
&= -\limsup_{n\to\infty} -s_n \checkmark
\end{aligned}$$

11.9

Suppose s_n is a sequence in $[a, b]$ that converges to s. We need to show that s is in $[a, b]$

We know $s_n \geq a$ for all n, but then, by the result of 8.9(a), this implies that $s \geq a$

Similarly, $s_n \leq b$ for all n, but then, by 8.9(b), this implies $s \leq b$

Date: Friday, October 8, 2021.

Therefore $a \leq s \leq b$, so s is in $[a, b]$ ✓

For part (b), the answer is **NO**, because $(0, 1)$ is not a closed set: For example, $t_n = \frac{1}{n}$ (with $n \geq 2$) is a sequence in $(0, 1)$ but whose limit is $0 \notin (0, 1)$.

11.11

First of all, if $\sup(S) \in S$, then we're done: Let $s_n \equiv \sup(S)$, then s_n is nondecreasing and converges to $\sup(S)$.

So, from now on, assume $\sup(S) \notin S$

Goal: Inductively construct an increasing sequence (s_n) such that for all n, we have

$$\sup(S) - \frac{1}{n} < s_n < \sup(S)$$

Then the squeeze theorem implies that $s_n \to \sup(S)$, and we would be done.

Base Case: Notice $\sup(S) - 1 < \sup(S)$, so by definition of sup, there is s_1 with $s_1 > \sup(S) - 1$. Moreover, by definition of $\sup(S)$ and the fact that $\sup(S) \notin S$, we get $s_1 < \sup(S)$, so $\sup(S) - 1 < s_1 < \sup(S)$

Inductive Step: Suppose we constructed $s_1 < s_2 < \cdots < s_n$ with $\sup(S) - \frac{1}{k} < s_k < \sup(S)$ for all $k = 1, \ldots, n$.

Consider $M = \max\left\{s_n, \sup(S) - \frac{1}{n+1}\right\} < \sup(S)$

Then by definition of $\sup(S)$ there is $s_{n+1} \in S$ such that $s_{n+1} > M$.

By definition of M this implies $s_{n+1} > s_n$ ✓and $s_{n+1} > \sup(S) - \frac{1}{n+1}$

Moreover, by definition of $\sup(S)$ and the fact that $\sup(S) \notin S$, we get $s_{n+1} < \sup(S)$, so $\sup(S) - \frac{1}{n+1} < s_{n+1} < \sup(S)$ ✓ □

12.4

STEP 1: Let N be given. First, let's show that

$$\sup\{s_n + t_n \mid n > N\} \leq \sup\{s_n \mid n > N\} + \sup\{t_n \mid n > N\}$$

But if $n > N$, then by definition of sup,

$$s_n \leq \sup\{s_n \mid n > N\}$$

$$\text{And} \quad t_n \leq \sup\{t_n \mid n > N\}$$

So adding both sides, we get

$$s_n + t_n \leq \sup\{s_n \mid n > N\} + \sup\{t_n \mid n > N\}$$

And taking the sup over all $n > N$, we get:

$$\sup\{s_n + t_n \mid n > N\} \leq \sup\{s_n \mid n > N\} + \sup\{t_n \mid n > N\}$$

STEP 2: Taking $\lim_{N\to\infty}$ in the above identity, we get:

$$\begin{aligned}
\limsup_{n\to\infty} s_n + t_n &\overset{DEF}{=} \lim_{N\to\infty} \sup\{s_n + t_n \mid n > N\} \\
&\overset{STEP1}{\leq} \lim_{N\to\infty} \sup\{s_n \mid n > N\} + \sup\{t_n \mid n > N\} \\
&= \lim_{N\to\infty} \sup\{s_n \mid n > N\} + \lim_{N\to\infty} \sup\{t_n \mid n > N\} \\
&= \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n \checkmark
\end{aligned}$$