Additional Problem 1: Apply Parseval to $f(x) = |x|$ on $[-\pi, \pi]$ to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-inx} dx = \frac{(-1)^{n+1}}{in}$$

and $\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} x dx = 0$

$$\hat{f}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| \, dx$$

$$= \frac{1}{2\pi}\left(\int_{-\pi}^{0} -x dx + \int_{0}^{\pi} x dx\right)$$

$$= \frac{1}{2\pi}\left(\int_{0}^{\pi} x dx - \int_{-\pi}^{0} x dx\right)$$

$$= \frac{1}{2\pi}\left(\frac{x^2}{2}\Big|_{0}^{\pi} - \frac{x^2}{2}\Big|_{-\pi}^{0}\right)$$

$$= \frac{1}{2\pi} \pi^2 = \frac{\pi}{2}$$

$$\hat{f}(n) = \frac{1}{2\pi}\left(\int_{-\pi}^{0} -x \cdot e^{-inx} dx + \int_{0}^{\pi} x \cdot e^{-inx} dx\right)$$

$$= \frac{1}{2\pi}\left(\int_{0}^{-\pi} x e^{-inx} dx + \int_{0}^{\pi} x \cdot e^{-inx} dx\right)$$

$$= \frac{1}{2\pi} \frac{1}{(-in)^2}\left((-inx-1)e^{-inx}\Big|_{0}^{-\pi} + (-inx-1)e^{-inx}\Big|_{0}^{\pi}\right)$$

$$= \frac{1}{2\pi} \frac{1}{-n^2}\left((in\pi-1)e^{in\pi}+1 + (-in\pi-1)e^{-in\pi}+1\right)$$

$$= \frac{1}{2\pi} \frac{1}{-n^2}\left(-2e^{in\pi}+2\right)$$ because $e^{in\pi} = e^{-in\pi} = \cos n\pi$

$$= \frac{1}{-n^2\pi}\left((-1)^n + 1\right)$$ Because

$$e^{in\pi} = \begin{cases} 1 & \text{when } n \text{ even} \\ -1 & \text{when } n \text{ odd.} \end{cases}$$

Then by

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx$$

we have: $|a_n| = \left|\hat{f}(n)e^{-inx}\right| = |\hat{f}(n)|$

So: $\text{L.H.S} = \sum_{n=-\infty}^{\infty} |a_n|^2$

$$= |a_0|^2 + \sum_{n\neq 0} |a_n|^2$$

$$= \left(\frac{\pi}{2}\right)^2 + \sum_{n\neq 0}\left|\frac{1}{-n^2\pi}\left((-1)^n+1\right)\right|^2$$

$$= \left(\frac{\pi}{2}\right)^2 + \frac{1}{\pi^2}\sum_{-\infty}^{+\infty}\left(\frac{2}{(2k+1)^2}\right)^2 \qquad k\in\mathbb{N}$$

symmetric

$$= \left(\frac{\pi}{2}\right)^2 + \frac{2}{\pi^2}\sum_{k=0}^{\infty}\frac{4}{(2k+1)^4}$$

$$= \frac{\pi^2}{4} + \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}$$

$$\text{R.H.S.} = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{1}{2\pi}\left.\frac{x^3}{3}\right|_{-\pi}^{\pi} = \frac{1}{2\pi}\cdot\frac{2\pi^3}{3} = \frac{\pi^2}{3}$$

$\because$ L.H.S = R.H.S.

we have $\frac{\pi^2}{4} + \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = \frac{\pi^2}{3}$

$$\frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = \frac{\pi^2}{12}$$

$$\Rightarrow \sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = \frac{\pi^4}{96}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} + \sum_{k=1}^{\infty}\frac{1}{(2k)^4} = \frac{\pi^4}{96} + \frac{1}{16}\sum_{k=1}^{\infty}\frac{1}{k^4}$$

Because $\sum_{n=1}^{\infty}\frac{1}{n^4}$ converges, Let $\sum_{n=1}^{\infty}\frac{1}{n^4} = a$

we have $a = \frac{\pi^4}{96} + \frac{1}{16}a \Rightarrow \frac{15}{16}a = \frac{\pi^4}{96} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = a = \frac{\pi^4}{90}$

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Additional Problem 2: Use the Riemann-Lebesgue Lemma to show

$$\int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}$$

We have $\int_{-\pi}^{\pi} D_N(t)\,dt = 2\pi$ So $\int_{-\pi}^{\pi} \frac{\sin(N+\frac{1}{2})x}{\sin(x/2)}\,dx = 2\pi$

$$2\pi = \int_{-\pi}^{\pi} \left[\left(\frac{1}{\sin(\frac{x}{2})} - \frac{2}{x}\right) + \frac{2}{x}\right] \sin(N+\tfrac{1}{2})x\,dx$$

$$= \int_{-\pi}^{\pi} \left(\frac{1}{\sin(\frac{x}{2})} - \frac{2}{x}\right)\sin(N+\tfrac{1}{2})x\,dx + \int_{-\pi}^{\pi} \frac{2}{x}\sin(N+\tfrac{1}{2})x\,dx.$$

Denoted as

$$= \mathrm{I} + \mathrm{II}$$

I: $\lim_{x\to 0} \left(\frac{1}{\sin\frac{x}{2}} - \frac{2}{x}\right) = 0$, So it has a removable point at 0

it is 2π periodic and integrable.

Thus By Lebesgue-Reimann Lemma.

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \left(\frac{1}{\sin\frac{x}{2}} - \frac{2}{x}\right)\sin nx = 0$$ thus,

By hint, we are allowed to ingore the "$+\frac{1}{2}$", then we have

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \left(\frac{1}{\sin\frac{x}{2}} - \frac{2}{x}\right)\sin(N+\tfrac{1}{2})x = 0$$

So $2\pi = \lim_{n\to\infty} 2\pi = \underbrace{\lim_{N\to\infty} \mathrm{I}}_{=0} + \lim_{N\to\infty}\mathrm{II} = \lim_{N\to\infty}\mathrm{II} = \int_{-\pi}^{\pi} \frac{2}{x}\sin(N+\tfrac{1}{2})x\,dx$

Then we have

$$\lim_{N\to\infty} \int_0^{\pi} \frac{2}{x}\sin(N+\tfrac{1}{2})x\,dx = \pi$$

$$\lim_{N\to\infty} \int_0^{\pi} \frac{\sin(N+\frac{1}{2})x}{x}\,dx = \frac{\pi}{2}$$

$$\lim_{N\to\infty} \int_0^{\pi} \frac{\sin(N+\frac{1}{2})x}{(N+\frac{1}{2})x}\cdot d(N+\tfrac{1}{2})x = \frac{\pi}{2}$$ By u-sub.

$$\lim_{N\to\infty} \int_0^{(N+\frac{1}{2})\pi} \frac{\sin y}{y}\,dy = \frac{\pi}{2}$$

By Mean Value theorem

$$\int_{N\pi}^{(N+\frac{1}{2})\pi} \frac{\sin y}{y} dy = \frac{1}{2}\pi \frac{\sin z}{z} \quad z \in [N, N+\frac{\pi}{2})$$

thus $\lim_{n\to\infty} \int_{N\pi}^{(N+\frac{1}{2})\pi} \frac{\sin y}{y} dy \to 0$ as $N \to \infty$

Then we have

$$\int_0^{N\pi} \frac{\sin y}{y} dy \to \frac{\pi}{2} \text{ as } N \to \infty$$

By MVT

$$\int_{N\pi}^{N+1\pi} \left| \frac{\sin y}{y} \right| dy \leq \pi \cdot \frac{1}{N\pi} = \frac{1}{N}$$

for any $t \in [N\pi, N+1\pi]$

$$\left| \int_{N\pi}^{t} \frac{\sin y}{y} dy \right| \leq \frac{1}{N} \to 0 \quad \text{as } N \to \infty$$

So

$$\int_0^{N\pi} \frac{\sin y}{y} dy - \left| \int_{N\pi}^{t} \frac{\sin y}{y} dy \right| \leq \int_0^t \frac{\sin y}{y} dy \leq \int_0^{N\pi} \frac{\sin y}{y} dy + \left| \int_{N\pi}^{t} \frac{\sin y}{y} dy \right|$$

$$\int_0^{N\pi} \frac{\sin y}{y} dy - \frac{1}{N} \leq \int_0^t \frac{\sin y}{y} dy \leq \int_0^{N\pi} \frac{\sin y}{y} dy + \frac{1}{N}$$

Then, because $t \in [N\pi, N+1\pi]$, take $N \to \infty$, $t \to \infty$ as well.

$$\lim_{N\to\infty} \int_0^{N\pi} \frac{\sin y}{y} dy - 0 \leq \lim_{t\to\infty} \int_0^{t} \frac{\sin y}{y} dy \leq \lim_{N\to\infty} \int_0^{N\pi} \frac{\sin y}{y} dy + 0$$

Thus we have:

$$\int_0^{\infty} \frac{\sin y}{y} dy = \lim_{N\to\infty} \int_0^{N\pi} \frac{\sin y}{y} dy = \frac{\pi}{2}$$

Additional Problem 3: Show that if f is 2π periodic and continuously differentiable, then the Fourier series of f is absolutely convergent

$\sum_{n=-\infty}^{+\infty} |\hat{f}(n)|$ Because $\widehat{f'}(n) = in\hat{f}(n)$.

$= \hat{f}(0) + \sum_{n\neq 0} \frac{1}{|n|}(\hat{f}'(n))$ Because Cauchy-Schwarz

$\leq \hat{f}(0) + \left(2\sum_{n=1}^{\infty} \frac{1}{|n|^2}\right)^{\frac{1}{2}} \left(\sum_{n\neq 0} (\hat{f}'(n))^2\right)^{\frac{1}{2}}$

By $\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx$ and $\sum_{n=1}^{\infty} \frac{1}{|n|^2} = \frac{\pi^2}{6}$ given in lecture.

$\leq \hat{f}(0) + \left(2\cdot\frac{\pi^2}{6}\right)^{\frac{1}{2}} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(x)|^2 dx - |\hat{f}(0)|^2\right)^{\frac{1}{2}}$

Because $f'(x)$ continuous. So it is integrable.

Thus above equation convergence.

i.e.:

$\sum_{-\infty}^{+\infty} |\hat{f}(n)| < \infty$, Fourier series of f is uniformly convergent.

Additional Problem 4: Show that if $f \in \mathcal{S}$, then $\hat{f}(\xi)$ is continuous on $\mathbb{R}$

$\xi, \varepsilon, N, M, \delta_0, \delta$ are fixed in the proof

Take any ξ in $\mathbb{R}$,

Given $\varepsilon > 0$, there is N such that $\int_{|x|>N} |f(x)|\,dx < \frac{\varepsilon}{2}$

Because $f \in \mathcal{S}$, f is bounded in $[-N, N]$, denoted $\max\limits_{[-N,N]} f(x) := M$

For all x in $[-N, N]$, $\lim\limits_{h \to 0} xh = 0$ and $\lim\limits_{xh \to 0} e^{-2\pi i xh} = 1$

x in $[-N, N]$

for some ε, there is $\delta_0 > 0$ s.t. for any $|xh| < \delta_0$

$$|e^{-2\pi i xh} - 1| < \min\{1, \frac{\varepsilon}{4NM}\}$$

Then there is $\delta = \frac{\delta_0}{N}$, for any $|h| < \delta$, we have $|xh| < N|h| < N\delta = \delta_0$, thus

$$|e^{-2\pi i xh} - 1| < \min\{1, \frac{\varepsilon}{4NM}\}$$

Take any h, which $|h| < \delta$

$$|\hat{f}(\xi+h) - \hat{f}(\xi)| = \left| \int_{-\infty}^{+\infty} f(x) e^{-2\pi i x(\xi+h)}\,dx - \int_{-\infty}^{+\infty} f(x) e^{-2\pi i x\xi}\,dx \right|$$

$$= \left| \int_{-\infty}^{+\infty} f(x) e^{-2\pi i x\xi} (e^{-2\pi i xh} - 1) \right|$$

$$\leq \left| \int_{|x|>N} f(x) e^{-2\pi i x\xi} (e^{-2\pi i xh} - 1) \right| + \left| \int_{-N}^{N} f(x) e^{-2\pi i x\xi} (e^{-2\pi i xh} - 1) \right|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ Because:}$$

$$\left| \int_{|x|>N} f(x) e^{-2\pi i x\xi} (e^{h} - 1)\,dx \right| \leq \int_{|x|>N} |f(x)|\, |e^{-2\pi i x\xi}|\, |e^{2\pi i xh} - 1|\,dx$$

$$\leq \int_{|x|>N} |f(x)|\,dx < \frac{\varepsilon}{2}$$

$$\left| \int_{-N}^{N} f(x) e^{-2\pi i x\xi} (e^{h} - 1)\,dx \right| \leq \int_{-N}^{N} |f(x)|\, |e^{-2\pi i x\xi}|\, |e^{2\pi i xh} - 1|\,dx$$

$$\leq \int_{-N}^{N} |f(x)|\, |e^{2\pi i xh} - 1|\,dx$$

$$\leq 2N \cdot M \frac{\varepsilon}{4NM} < \frac{\varepsilon}{2}$$

In Summary

For Any ξ in $\mathbb{R}$, For every $\varepsilon > 0$, exists $\delta > 0$, such that for any $|h| < \delta$

$$|\hat{f}(\xi+h) - \hat{f}(\xi)| < \varepsilon$$

means $\hat{f}(\xi)$ continuous on $\mathbb{R}$

Additional Problem 5: Prove the **Poisson Summation Formula**, that is, if $f \in \mathcal{S}$ then (see hints)

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$$

let $F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$

$F(x)$ has following Fourier series representation:

$$F(x) = \sum_{k\in\mathbb{Z}} \hat{F}(k) e^{2\pi i k x} \quad (*)$$

$\hat{F}(k)$ are $F(x)$'s Fourier coefficients

$$\hat{F}(k) = \int_0^1 F(x) \cdot e^{-2\pi i k x} dx$$ ← 1 periodic

$$= \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n) e^{-2\pi i k x} dx$$

$$= \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n) e^{-2\pi i k x} dx$$

Because $f \in S$, we can exchange it

$$= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(x) e^{-2\pi i k x} dx$$

$$= \int_{\mathbb{R}} f(x) e^{-2\pi i k x} dx$$, it is exactly the Fourier transform of $f(x)$

$$= \hat{f}(k)$$

Then: By $(*)$, we have

$$F(x) = \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{2\pi i k x}$$, which ends the proof