MATH S4062 – HOMEWORK 5

Additional Problem 1: Apply Parseval to $f(x) = |x|$ on $[-\pi, \pi]$ to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Additional Problem 2: Use the Riemann-Lebesgue Lemma to show

$$\int_0^{\infty} \frac{\sin(x)}{x} dx = \frac{\pi}{2}$$

Additional Problem 3: Show that if f is 2π periodic and continuously differentiable, then the Fourier series of f is absolutely convergent

Additional Problem 4: Show that if $f \in \mathcal{S}$, then $\hat{f}(\xi)$ is continuous on $\mathbb{R}$

Additional Problem 5: Prove the **Poisson Summation Formula**, that is, if $f \in \mathcal{S}$ then (see hints)

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$$

Aside: In particular, if you let $x = 0$ in the above, you get

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)$$

Date: Due: Friday, July 22, 2022.

Hints:

Additional Problem 1: Since f is even, the Fourier series is just a cosine series

Additional Problem 2: Start with the fact that $\int_{-\pi}^{\pi} D_N(x)dx = 2\pi$ where D_N is the Dirichlet kernel.

Then use the explicit formula of D_N in terms of sin.

You can then write the denominator of D_N as:

$$\frac{1}{\sin\left(\frac{x}{2}\right)} = \left(\frac{1}{\sin\left(\frac{x}{2}\right)} - \frac{2}{x}\right) + \frac{2}{x}$$

This gives you two integrals which add up to 2π.

For the first integral, use the Riemann-Lebesgue Lemma to show that the limit as $N \to \infty$ of the integral is 0. You can ignore the $+\frac{1}{2}$ if you want, and you're allowed to use familiar limit rules from Calculus.

For the second integral, use the fact that it's an even function to get an integral from 0 to π and finally use the $u-$sub $u = \left(N + \frac{1}{2}\right)x$ and take $N \to \infty$

Additional Problem 3: First apply Parseval to the Fourier coefficients of $f'(x)$. We have shown in lecture that $\widehat{f'}(n) = in\hat{f}(n)$. Then write $\hat{f}(n) = \frac{1}{n}\left(n\hat{f}(n)\right)$ and use the Cauchy-Schwarz inequality for series:

$$\left(\sum_{n=-\infty}^{\infty} a_n b_n\right)^2 \leq \left(\sum_{n=-\infty}^{\infty} (a_n)^2\right)\left(\sum_{n=-\infty}^{\infty} (b_n)^2\right)$$

Additional Problem 4: Given $\epsilon > 0$, since f is Schwartz, there is N such that $\int_{|x|>N} |f(x)|\, dx < \frac{\epsilon}{2}$ (no need to prove this) Then calculate $\hat{f}(\xi+h) - \hat{f}(\xi)$ and split up the resulting integral into two regions, one where $|x| > N$ and one where $|x| \leq N$. And then remember that h is small

Additional Problem 5: **Beware:** Since f is not periodic, $\hat{f}(n)$ is defined as

$$\hat{f}(n) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x} dx$$

(The Fourier transform at the integer n)

That said, notice that both the left-hand-side and the right-hand-side are $1-$periodic and continuous since f is Schwartz (no need to show this) and so it's enough by uniqueness to show that for all m, the $m-$th Fourier coefficient of the left hand side is equal to the one of right-hand-side. Here the $m-$th Fourier coefficient of a periodic function g is defined as $\int_0^1 g(x) e^{-2\pi i m x} dx$

You're allowed to interchange the sum and integral without proof (which follows since f is Schwartz)