MATH 409 – HOMEWORK 6

Reading: Sections 12 and 14

- **Section 12:** 8, 12 (See Note), 14, AP1, AP2
- **Section 14:** 6(a), AP3, AP4, AP5, AP6

Note: If by any chance you're using the first edition of the textbook, for Problem 12, you must *also* do part (c), which is:

(c) Give an example where $\lim \sigma_n$ exists but $\lim s_n$ does not exist.

Additional Problem 1: The results in section 12 are strict:

(a) Give an example of a sequence (s_n) which converges to a negative real number $s < 0$ and a sequence (t_n) such that

$$\limsup s_n t_n \neq (\limsup s_n)(\limsup t_n)$$

(b) Give an example of sequences (s_n) and (t_n) such that

$$\limsup s_n + t_n \neq (\limsup s_n) + (\limsup t_n)$$

Additional Problem 2: Mimic the proof of the Limsup Product Rule to show: if (s_n) converges to s and (t_n) is any bounded sequence, then

$$\limsup_{n\to\infty} s_n + t_n = \left(\limsup_{n\to\infty} s_n\right) + \left(\limsup_{n\to\infty} t_n\right)$$

Additional Problem 3: Determine which of the following series converge. Justify your answers:

Date: Due: Friday, October 15, 2021.

(a) $\sum \frac{n}{2^n}$ (use the ratio test)

(b) $\sum \frac{n}{2^n}$ (this time use the root test)

(c) $\sum \frac{n!}{n^n}$

(d) $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ (You may use $\ln(n) < n$)

(e) $\sum \left(\frac{5}{(-1)^n+4}\right)^n$

Additional Problem 4: Find the sum of the following series:

(a) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (you may assume that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$)

(b) $\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}}$ (use $\frac{n-1}{2^{n+1}} = \frac{n}{2^n} - \frac{n+1}{2^{n+1}}$)

(c) $\sum_{n=1}^{\infty} \frac{n}{2^n}$ (use (b))

(d) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ (You may assume that $\frac{1}{n(n+1)(n+2)} = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2n+4}$). Look at the diagonal terms. If you're completely stuck, check out: Diagonally Telescoping Sum)

Additional Problem 5: There's another neat convergence test called the block test (because it groups the terms in blocks of length 2^k):

Block Test:

If (a_n) is a sequence of non-negative terms with $a_1 \geq a_2 \geq a_3 \dots$ then $\sum a_n$ converges if and only if the following series converges:

$$\sum_{k=0}^{\infty} 2^k \, a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \dots$$

For parts (a) and (b), use the block test to figure out for which p the following series converge. For parts (c) and (d), figure out if the series converges. Do **NOT** use the Integral Test!

(a) $\sum_{n=1}^{\infty} \frac{1}{n^p}$

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$ (Use (a); here k in the block test starts at 1)

(c) $\sum \frac{1}{n\ln(n)(\ln[\ln(n)])}$ (use $\ln(2) \leq \ln(k)$ and (b) with $p = 1$)

(d) $\sum \frac{1}{n\ln(n)(\ln[\ln(n)])^2}$ (use $\ln(2) + \ln(k) \geq \ln(k)$ and (b) with $p = 2$)

Additional Problem 6:

(a) Prove that the following series converges: $\sum \frac{1}{n!}$

Definition:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

(b) Use the definition of e (and the hints at the end) to show

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$$

Hints:

12.8 Start with $s_n \leq \sup\{s_n \mid n > N\}$, and similar for t_n, whenever $n > N$.

12.12(a) You just need to show the third inequality, since the middle one is obvious, and the first one is similar to the third one. Then prove the hint given in the book.

For this, write $s_1 + \cdots + s_n = s_1 + \cdots + s_N + (s_{N+1} + \cdots + s_n)$, and notice that each term in the parenthesis is bounded above by $\sup\{s_n \mid n > N\}$. At some point use if $n > M$ then $\frac{1}{n} < \frac{1}{M}$, to make the right-hand-side independent of n, and finally take the sup of the left-hand-side over $n > M$.

Then be careful: *First* let $M \to \infty$ and *then* let $N \to \infty$.

12.14 Use the Pre-Ratio Test. For 12.14(b), let $s_n = \frac{n!}{n^n}$. After simplifying, at some point you'll need to factor out n from the top and bottom. You'll also need to use the fact that $\left(1 + \frac{1}{n}\right)^n \to e$

AP 6(b) Define

$$s_n = \sum_{k=0}^{n} \frac{1}{k!} \text{ and } t_n = \left(1 + \frac{1}{n}\right)^n$$

Recall the Binomial Theorem, which says:

Binomial Theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

With

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-(k-1))}{k!}$$

Use the binomial theorem to show that

$$t_n = \sum_{k=0}^{n} \frac{1}{k!}\left(1-\frac{1}{n}\right)\dots\left(1-\left(\frac{k-1}{n}\right)\right)$$

And conclude that $t_n \leq s_n$ (notice each term in the parenthesis is ≤ 1) Now take $\limsup_{n\to\infty}$ on both sides.

On the other hand, if m is fixed and $n \geq m$, notice that we have:

$$\begin{aligned} t_n &= \sum_{k=0}^{n} \frac{1}{k!}\left(1-\frac{1}{n}\right)\dots\left(1-\left(\frac{k-1}{n}\right)\right) \\ &\geq \sum_{k=0}^{m} \frac{1}{k!}\left(1-\frac{1}{n}\right)\dots\left(1-\left(\frac{k-1}{n}\right)\right) \end{aligned}$$

Now take $\liminf_{n\to\infty}$ on both sides (keeping in mind m is fixed, so each term in the parenthesis goes to 1), and then take $\lim_{m\to\infty}$ of the right-hand-side

Finally conclude by the limsup squeeze theorem.