HOMEWORK 6 – AP SOLUTIONS

AP 1

(a) Let $s_n = -1$ for all n (which converges to -1) and let $t_n = (-1)^n$. Then

$$\limsup_{n\to\infty} s_n = -1 \quad \limsup_{n\to\infty} t_n = 1$$

But:

$$\begin{aligned}\limsup_{n\to\infty} s_n t_n &= \limsup_{n\to\infty} (-1)(-1)^n \\ &= \limsup_{n\to\infty} (-1)^{n+1} \\ &= 1 \\ &\neq \left(\limsup_{n\to\infty} s_n\right)\left(\limsup_{n\to\infty} t_n\right) = (-1)(1) = -1\end{aligned}$$

(b) Let $s_n = (-1)^n$ and $t_n = -s_n = (-1)^{n+1}$, then

$$\limsup_{n\to\infty} s_n + t_n = \limsup_{n\to\infty} (-1)^n - (-1)^n = \limsup_{n\to\infty} 0 = 0$$

But:

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$$\limsup_{n\to\infty} s_n + \limsup_{n\to\infty} s_n = \limsup_{n\to\infty}(-1)^n + \limsup_{n\to\infty}(-1)^{n+1} = 1 + 1 = 2$$

AP 2

First of all, in 12.4, we've shown that for any sequences (s_n) and (t_n), we have:

$$\limsup_{n\to\infty} s_n + t_n \leq \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n$$

So it is enough to show

$$\limsup_{n\to\infty} s_n + t_n \geq \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n$$

Let $t = \limsup_{n\to\infty} t_n$. Notice t is finite since (t_n) is bounded.

Then let (t_{n_k}) be a subsequence of (t_n) converging to t.

Then $(s_{n_k}+t_{n_k})$ is a subsequence of (s_n+t_n) converging to $s+t$, and so, since $\limsup_{n\to\infty} s_n + t_n$ is the largest subsequential limit of $(s_n + t_n)$, we have:

$$\limsup_{n\to\infty} s_n+t_n \geq s+t = \lim_{n\to\infty} s_n+\limsup_{n\to\infty} t_n = \limsup_{n\to\infty} s_n+\limsup_{n\to\infty} t_n \checkmark \quad \square$$

AP 3

(a)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}\right| = \left(\frac{2^n}{2^{n+1}}\right)\left(\frac{n+1}{n}\right) = \frac{1}{2}\left(\frac{n+1}{n}\right) \to \frac{1}{2}(1) = \frac{1}{2} < 1$$

Therefore $\sum \frac{n}{2^n}$ converges

(b)

$$|a_n|^{\frac{1}{n}} = \left|\frac{n}{2^n}\right|^{\frac{1}{n}} = \frac{n^{\frac{1}{n}}}{2} \to \frac{1}{2} < 1$$

Therefore $\sum \frac{n}{2^n}$ converges.

(c)

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \\
&= \frac{(n+1)!}{n!}\frac{n^n}{(n+1)^{n+1}} \\
&= (n+1)\frac{n^n}{(n+1)^{n+1}} \\
&= \frac{n^n}{(n+1)^n} \\
&= \left(\frac{n}{n+1}\right)^n \\
&= \left(\frac{n}{n\left(1+\frac{1}{n}\right)}\right)^n \\
&= \left(\frac{1}{1+\frac{1}{n}}\right)^n \\
&= \frac{1}{\left(1+\frac{1}{n}\right)^n} \\
&\to \frac{1}{e} < 1
\end{aligned}$$

Therefore $\sum \frac{n!}{n^n}$ converges.

(d) Since $\ln(n) < n$ we have $\frac{1}{\ln(n)} > \frac{1}{n}$, and therefore, since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, by comparison, $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges.

(e) Notice that, if n is even, then

$$a_n = \left(\frac{5}{(-1)^n + 4}\right)^{\frac{1}{n}} = \left(\frac{5}{5}\right)^{\frac{1}{n}} = 1$$

So since $a_n \not\to 0$ we get that $\sum \left(\frac{5}{(-1)^n+4}\right)^{\frac{1}{n}}$ diverges.

AP 4

(a)

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} \\ &= \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} \\ &= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cdots + \cancel{\frac{1}{n}} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1$$

(b) Following the hint, we get, for all n

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{k-1}{2^{k+1}} \\ &= \sum_{k=1}^{n} \frac{k}{2^k} - \frac{k+1}{2^{k+1}} \\ &= \frac{1}{2} - \frac{2}{4} + \frac{2}{4} - \frac{3}{8} + \frac{3}{8} - \frac{4}{16} + \cdots + \frac{n}{2^n} - \frac{n+1}{2^{n+1}} \\ &= \frac{1}{2} - \frac{n+1}{2^{n+1}} \\ &= \frac{1}{2} - \frac{1}{2}\left(\frac{n}{2^n}\right) - \frac{1}{2^{n+1}} \end{aligned}$$

However, since $\frac{n}{2^n} \to 0$ as n goes to infinity, we get

$$\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \lim_{n\to\infty} s_n = \frac{1}{2}$$

(c)

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \sum_{n=1}^{\infty} \frac{n-1+1}{2^n} = \sum_{n=1}^{\infty} \frac{n-1}{2^n} + \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + 1 = \frac{3}{2}$$

(d)

$$\begin{aligned}
s_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} \\
&= \sum_{k=1}^{n} \frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2k+4} \\
&= \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \frac{1}{6} \\
&\quad + \frac{1}{4} - \frac{1}{3} + \frac{1}{8} \\
&\quad + \frac{1}{6} - \frac{1}{4} + \frac{1}{10} \\
&\quad + \frac{1}{8} - \frac{1}{5} + \frac{1}{12} \\
&\quad \cdots \\
&\quad + \cancel{\frac{1}{2n}} - \cancel{\frac{1}{n+1}} + \frac{1}{2n+4} \\
&= \frac{1}{4} - \frac{1}{2} + \frac{1}{4} + \frac{1}{2n+4} \\
&= \frac{1}{4} + \frac{1}{2n+4}
\end{aligned}$$

Hence:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{4} + \frac{1}{2n+4} = \frac{1}{4}$$

AP 5

(a) Here $a_k = \frac{1}{k^p}$, so:

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = \sum_{k=0}^{\infty} \frac{2^k}{(2^k)^p} = \sum_{k=0}^{\infty} \left(\frac{2}{2^p}\right)^k = \sum_{k=0}^{\infty} \left(2^{1-p}\right)^k$$

Which is the geometric series, which converges if and only if $2^{1-p} < 1$, that is $1 - p < 0$, so $p > 1$

(b) Here $a_k = \frac{1}{k(\ln(k))^p}$, so

$$\sum_{k=1}^{\infty} 2^k a_{2^k} = \sum_{k=1}^{\infty} \frac{2^k}{2^k\left(\ln(2^k)\right)^p} = \sum_{k=1}^{\infty} \frac{1}{(k\ln(2))^p} = \frac{1}{(\ln(2))^p} \sum_{k=1}^{\infty} \frac{1}{k^p}$$

Which, by (a) converges if and only if $p > 1$

(c)

$$\begin{aligned}
\sum_{k=2}^{\infty} 2^k a_{2^k} &= \sum_{k=1}^{\infty} \frac{2^k}{2^k \ln(2^k) \ln(\ln(2^k))} \\
&= \sum_{k=2}^{\infty} \frac{1}{k\ln(2)\ln(k\ln(2))} \\
&= \frac{1}{\ln(2)} \sum_{k=1}^{\infty} \frac{1}{k\left(\ln(k) + \ln(2)\right)} \\
&\geq \frac{1}{\ln(2)} \sum_{k=1}^{\infty} \frac{1}{k\left(\ln(k) + \ln(k)\right)} \\
&= \frac{1}{\ln(2)} \sum_{k=1}^{\infty} \frac{1}{2k\ln(k)} \\
&= \frac{1}{2\ln(2)} \sum_{k=1}^{\infty} \frac{1}{2\ln(k)}
\end{aligned}$$

Which diverges by the result of (b) with $p = 1$, and therefore $\sum 2^k a_{2^k}$ diverges by the comparison test, so the original series diverges by the block test.

(d)

$$\begin{aligned}\sum_{k=2}^{\infty} 2^k a_{2^k} &= \sum_{k=1}^{\infty} \frac{2^k}{2^k \ln(2^k)\left(\ln(\ln(2^k))\right)} \\ &= \sum_{k=2}^{\infty} \frac{1}{k\ln(2)\left(\ln(k\ln(2))\right)^2} \\ &= \frac{1}{\ln(2)} \sum_{k=1}^{\infty} \frac{1}{k\left(\ln(k)+\ln(2)\right)^2} \\ &\leq \frac{1}{\ln(2)} \sum_{k=1}^{\infty} \frac{1}{k\left(\ln(k)\right)^2}\end{aligned}$$

Which converges by the result of (b) with $p = 2$, therefore $\sum 2^k a_{2^k}$ converges by the comparison test, so the original series converges by the block test.

AP 6

(a)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0 < 1$$

Hence $\sum \frac{1}{n!}$ converges

(b) **STEP 1:** Define (as in the hint)

$$s_n = \sum_{k=0}^{n} \frac{1}{k!} \text{ and } t_n = \left(1 + \frac{1}{n}\right)^n$$

Then, by the binomial theorem, we have:

$$t_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \left(\frac{1}{n}\right)^k = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n^k}\right)$$

But:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-(k-1))}{k!}$$

And therefore:

$$\begin{aligned}
\frac{\binom{n}{k}}{n^k} =& \frac{1}{k!}\left(\frac{n(n-1)\dots(n-(k-1))}{n^k}\right) \\
=& \frac{1}{k!}\left(\frac{n(n-1)\dots(n-(k-1))}{nn\dots n}\right) \\
=& \frac{1}{k!}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\dots\left(\frac{n-(k-1)}{n}\right) \\
=& \frac{1}{k!}\left(1 - \frac{1}{n}\right)\dots\left(1 - \left(\frac{k-1}{n}\right)\right)
\end{aligned}$$

Hence, by the above, we get

$$\begin{aligned} t_n &= \sum_{k=0}^{n} \frac{1}{k!} \underbrace{\left(1 - \frac{1}{n}\right)}_{\leq 1} \cdots \underbrace{\left(1 - \left(\frac{k-1}{n}\right)\right)}_{\leq 1} \\ &\leq \sum_{k=0}^{n} \frac{1}{k!} \\ &= s_n \end{aligned}$$

Now taking $\limsup$ on both sides, and remembering that $\lim_{n \to \infty} s_n = \sum_{k=0}^{\infty} \frac{1}{k!} = e$ (by definition), we get:

$$\limsup_{n \to \infty} t_n \leq \limsup_{n \to \infty} s_n = e$$

So $\limsup t_n \leq e$

STEP 2: On the other hand, if m is fixed and $n \geq m$, then we have:

$$\begin{aligned} t_n &= \sum_{k=0}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \left(\frac{k-1}{n}\right)\right) \\ &\geq \sum_{k=0}^{m} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \left(\frac{k-1}{n}\right)\right) \end{aligned}$$

Notice that, as $n \to \infty$, for each $k = 0, \ldots, m$ (fixed), we have $1 - \frac{1}{n} \to 1$, $\ldots 1 - \left(\frac{k-1}{n}\right) \to 1$, and therefore the limit as $n \to \infty$ of the right-hand-side is just $\sum_{k=0}^{m} \frac{1}{k!}$

Therefore, taking $\liminf$ on both sides, we get:

$$\liminf_{n\to\infty} t_n \geq \liminf_{n\to\infty} \sum_{k=0}^{m} \frac{1}{k!}\left(1-\frac{1}{n}\right)\dots\left(1-\left(\frac{k-1}{n}\right)\right) = \sum_{k=0}^{m} \frac{1}{k!} = s_m$$

Therefore $\liminf_{n\to\infty} t_n \geq s_m$, and since this is true for all m, we can let $m \to \infty$ to get

$$\liminf_{n\to\infty} t_n \geq \lim_{m\to\infty} s_m = e$$

Hence $\liminf_{n\to\infty} t_n = e$

Therefore $\liminf_{n\to\infty} t_n = \limsup_{n\to\infty} t_n = e$ and by the limsup squeeze theorem, we have:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} t_n = e$$