HOMEWORK 6 – SELECTED BOOK SOLUTIONS

12.8

STEP 1: Let N be given, then for all $n > N$, by definition of sup

$$s_n \leq \sup \{s_n \mid n > N\} \qquad t_n \leq \sup \{t_n \mid n > N\}$$

Therefore, since $s_n \geq 0$ and $t_n \geq 0$, we get:

$$s_n t_n \leq \sup \left(\{s_n \mid n > N\}\right) t_n \leq \left(\sup \{s_n \mid n > N\}\right) \sup \{t_n \mid n > N\}$$

Now taking the sup over $n > N$, we get:

$$\sup \{s_n t_n \mid n > N\} \leq \sup \{s_n \mid n > N\} \sup \{t_n \mid n > N\}$$

STEP 2: Now taking the limit as $N \to \infty$ in the above, we get

$$\begin{aligned}
\limsup_{n\to\infty} s_n t_n &\overset{DEF}{=} \lim_{N\to\infty} \sup \{s_n t_n \mid n > N\} \\
&\overset{STEP1}{\leq} \lim_{N\to\infty} \sup \{s_n \mid n > N\} \sup \{t_n \mid n > N\} \\
&= \lim_{N\to\infty} \sup \{s_n \mid n > N\} \lim_{N\to\infty} \sup \{t_n \mid n > N\} \\
&= \left(\limsup_{n\to\infty} s_n\right) \left(\limsup_{n\to\infty} t_n\right) \checkmark
\end{aligned}$$

Date: Friday, October 15, 2021.

12.12(a)

The middle inequality follows because lim inf ≤ lim sup, so let's first show the third inequality, which is:

$$\limsup_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} s_n$$

STEP 1: Let N be given, and suppose $M > N$. Let's show that

$$\sup \{\sigma_n \mid n > M\} \leq \frac{1}{M}(s_1 + \cdots + s_N) + \sup \{s_n \mid n > N\}$$

Notice that for all $n > M$, we have

$$\sigma_n = \frac{1}{n}(s_1 + \cdots + s_n) = \frac{s_1 + \cdots + s_N}{n} + \frac{s_{N+1} + \cdots + s_n}{n}$$

Notice there are $n - (N+1) + 1 = n - N$ terms in $s_{N+1} + \cdots + s_n$. Moreover, by definition of sup, each term $s_{N+1}, s_{N+2}, \ldots, s_n$ is $\leq \sup \{s_n \mid n > N\}$, and therefore we get:

$$\begin{aligned}\sigma_n \leq& \frac{s_1 + \cdots + s_N}{n} + \frac{n-N}{n} \sup \{s_n \mid n > N\} \\ \leq& \frac{s_1 + \cdots + s_N}{M} + \sup \{s_n \mid n > N\}\end{aligned}$$

(Here we used $n > M \Rightarrow \frac{1}{n} < \frac{1}{M}$)

Since the right-hand-side doesn't depend on n, we therefore get:

$$\sup \{\sigma_n \mid n > M\} \leq \frac{s_1 + \cdots + s_N}{M} + \sup \{s_n \mid n > N\} \checkmark$$

STEP 2: Now be careful: *First* let $M \to \infty$ to get:

$$\begin{aligned}\limsup_{n\to\infty} \sigma_n &= \lim_{M\to\infty} \sup\{\sigma_n \mid n > M\} \\ &\leq \lim_{M\to\infty} \frac{1}{M}(s_1 + s_2 + \cdots + s_N) + \sup\{s_n \mid n > N\} \\ &= \lim_{M\to\infty} \frac{1}{M}(s_1 + s_2 + \cdots + s_N) + \lim_{M\to\infty} \sup\{s_n \mid n > N\}\end{aligned}$$

But since N is fixed, $s_1 + s_2 + \cdots + s_N$ doesn't depend on M, and so

$$\lim_{M\to\infty} \frac{1}{M}(s_1 + s_2 + \cdots + s_N) = 0$$

And moreover $\{s_n \mid n > N\}$ is constant with respect to M, and so

$$\lim_{M\to\infty} \sup\{s_n \mid n > N\} = \sup\{s_n \mid n > N\}$$

Therefore we get:

$$\limsup_{n\to\infty} \sigma_n \leq \sup\{s_n \mid n > N\}$$

But since the left-hand-side doesn't depend on N, we can let N go to ∞ to get:

$$\limsup_{n\to\infty} \sigma_n \leq \lim_{N\to\infty} \sup\{s_n \mid n > N\} = \limsup_{N\to\infty} s_n \checkmark$$

STEP 3: Now let's show

$$\liminf_{n\to\infty} \sigma_n \geq \liminf_{n\to\infty} s_n$$

As before, let N be given and suppose $M > N$, then if $n > M$, we get

$$\begin{aligned}
\sigma_n =& \frac{1}{n}\left(s_1 + \cdots + s_n\right) \\
=& \underbrace{\frac{s_1 + \cdots + s_N}{n}}_{\geq 0} + \frac{s_{N+1} + \cdots + s_n}{n} \\
\geq& \frac{s_{N+1} + \cdots + s_n}{n} \\
\geq& \left(\frac{n-N}{n}\right) \inf\left\{s_n \mid n > N\right\} \\
=& \left(1 - \frac{N}{n}\right) \inf\left\{s_n \mid n > N\right\} \\
\geq& \left(1 - \frac{N}{M}\right) \inf\left\{s_n \mid n > N\right\}
\end{aligned}$$

But since the right-hand-side doesn't depend on n, we can take inf of the left side over $n > M$ to get:

$$\inf\left\{\sigma_n \mid n > M\right\} \geq \left(1 - \frac{N}{M}\right) \inf\left\{s_n \mid n > N\right\}$$

And taking $M \to \infty$, we get

$$\begin{aligned}
\liminf_{n\to\infty} \sigma_n =& \lim_{M\to\infty} \inf\left\{\sigma_n \mid n > M\right\} \\
\geq& \lim_{M\to\infty} \left(1 - \frac{N}{M}\right) \inf\left\{s_n \mid n > N\right\} \\
=& \inf\left\{s_n \mid n > N\right\}
\end{aligned}$$

Since the left hand side doesn't depend on N, can take $N \to \infty$ on the right hand side, we get:

$$\liminf_{n\to\infty} \sigma_n \geq \lim_{N\to\infty} \inf\left\{s_n \mid n > N\right\} = \liminf_{n\to\infty} s_n \checkmark$$

12.12(b)

If $\lim_{n\to\infty} s_n = L$, then we get:

$$L = \liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} s_n = L$$

Therefore

$$\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = L$$

So by the $\limsup$ squeeze theorem, we have

$$\lim_{n\to\infty} s_n = L\checkmark$$

12.12(c)

Let $(s_n) = (-1)^n$, then $\lim_{n\to\infty} s_n$ doesn't exist, but

$$\begin{aligned}\sigma_n =& \frac{1}{n}\left(s_1 + s_2 + \cdots + s_n\right)\\ =& \frac{1}{n}\left(-1+1-1+1-1+1+\cdots+(-1)^n\right)\\ =& \frac{a_n}{n}\end{aligned}$$

Where a_n is either 0 (if n is even) or 1 if n is odd, but then by the squeeze theorem, we get that $\sigma_n \to 0$ as $n \to \infty$ ✓

Note: Strictly speaking we don't have $s_n \geq 0$ here, so if you want to modify this, you can simply let $s_n = \frac{(-1)^n+1}{2} = 0, 1, 0, 1, \ldots$, then s_n doesn't converge, but σ_n converges to $\frac{1}{2}$

12.14(a)

Let $s_n = n!$, then:

$$\left|\frac{s_{n+1}}{s_n}\right| = \frac{(n+1)!}{n!} = n+1$$

Therefore:

$$\liminf_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| = \lim_{n\to\infty} n+1 = \infty$$

So by the Pre-Root test, we have:

$$\liminf_{n\to\infty}(n!)^{\frac{1}{n}} = \liminf_{n\to\infty}|s_n|^{\frac{1}{n}} \geq \liminf_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| = \infty$$

Therefore

$$\lim_{n\to\infty}(n!)^{\frac{1}{n}} = \infty$$

12.14(b)

First of all,

$$\frac{1}{n}(n!)^{\frac{1}{n}} = \left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$$

So let $s_n = \frac{n!}{n^n}$, then:

$$\begin{aligned}
\left|\frac{s_{n+1}}{s_n}\right| =& \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \\
=& \frac{(n+1)!}{n!}\frac{n^n}{(n+1)^{n+1}} \\
=& (n+1)\frac{n^n}{(n+1)^{n+1}} \\
=& \frac{n^n}{(n+1)^n} \\
=& \left(\frac{n}{n+1}\right)^n \\
=& \left(\frac{n}{n\left(1+\frac{1}{n}\right)}\right)^n \\
=& \left(\frac{1}{1+\frac{1}{n}}\right)^n \\
=& \frac{1}{\left(1+\frac{1}{n}\right)^n} \\
\to& \frac{1}{e}
\end{aligned}$$

Therefore:

$$\lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| = \frac{1}{e}$$

So by the Corollary of the Pre-Root Test, we have

$$\lim_{n\to\infty}\frac{1}{n}(n!)^{\frac{1}{n}} = \lim_{n\to\infty}|s_n|^{\frac{1}{n}} = \frac{1}{e}$$

14.6(a)

Since (b_n) is bounded, there is M such that for all n, $|b_n| \leq M$

Since $\sum |a_n|$ converges, by the Cauchy criterion with $\frac{\epsilon}{M}$ there is N such that for all $n \geq m > N$, we have $\sum_{k=m}^{n} |a_k| < \frac{\epsilon}{M}$. But then, for the same N, if $n \geq m > N$, we have:

$$\left| \sum_{k=m}^{n} a_k b_k \right| \leq \sum_{k=m}^{n} |a_k b_k| = \sum_{k=m}^{n} |a_k| \, |b_k| \leq \sum_{k=m}^{n} |a_k| \, M = M \sum_{k=m}^{n} |a_k| < M \left(\frac{\epsilon}{M} \right) = \epsilon \checkmark$$

Hence, by the Cauchy criterion, $\sum a_n b_b$ converges.