6. If $f(0, 0) = 0$ and

$$f(x, y) = \frac{xy}{x^2 + y^2} \qquad \text{if } (x, y) \neq (0, 0),$$

prove that $(D_1 f)(x, y)$ and $(D_2 f)(x, y)$ exist at every point of R^2, although f is not continuous at $(0, 0)$.

1) when $(x,y) \neq (0,0)$ then $x^2+y^2 \neq 0$

$$(D_1 f)(x,y) = \frac{y(x^2+y^2) - 2x^2 y}{(x^2+y^2)^2} = \frac{y(y^2-x^2)}{(x^2+y^2)^2}$$

$$(D_2 f)(x,y) = \frac{x(x^2+y^2) - 2xy^2}{(x^2+y^2)^2} = \frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

2) when $(x,y) = (0,0)$

$$(D_1 f)(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \qquad (f(0,0)=0)$$

$$= \lim_{h\to 0} \frac{f(h,0)}{h} = 0 \quad \text{because } f(h,0) \equiv 0$$

Similarly

$$(D_2 f)(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(0,h)}{h} = 0$$

3) f is not continuous at $(0,0)$

$$f(x,x) = \frac{x^2}{x^2+x^2} = \frac{1}{2} \quad \text{for all } x \neq 0$$

hence $\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2} \neq 0 = f(0,0)$

thus $f(x,y)$ is not continuous at $(0,0)$

8. Suppose that f is a differentiable real function in an open set $E \subset R^n$, and that f has a local maximum at a point $\mathbf{x} \in E$. Prove that $f'(\mathbf{x}) = 0$.

Let y be any element of $\mathbb{R}^n$, and consider

$\varphi(t) = f(x+ty)$. Because f is $\mathbb{R}^n \to \mathbb{R}$, $t \in \mathbb{R}$, $\varphi(t): \mathbb{R} \to \mathbb{R}$

$\varphi(t)$ real function on $\mathbb{R} \to \mathbb{R}$

defined near $t=0$

$\varphi'(t) = f'(x+ty) \cdot y$ by chain rule

$\varphi(t)$ is differentiable $\mathbb{R} \to \mathbb{R}$ function

f has a local maximum at a point x_0

then $\varphi(t) = f(x_0+ty)$ has maximum on $t=0$

It follows that $\varphi'(0) = 0$,

then $\varphi'(0) = 0$

$\Rightarrow f'(x+0 \cdot y) \cdot y = 0$

$\Rightarrow f'(x) \cdot y = 0$

Because y is Arbitrary, So $f'(x)$ is the Zero linear Transformation.

14. Define $f(0, 0) = 0$ and

$$f(x, y) = \frac{x^3}{x^2 + y^2} \qquad \text{if } (x, y) \neq (0, 0).$$

(*a*) Prove that $D_1 f$ and $D_2 f$ are bounded functions in R^2. (Hence f is continuous.)
(*b*) Let $\mathbf{u}$ be any unit vector in R^2. Show that the directional derivative $(D_{\mathbf{u}} f)(0, 0)$ exists, and that its absolute value is at most 1.
(*c*) Let γ be a differentiable mapping of R^1 into R^2 (in other words, γ is a differentiable curve in R^2), with $\gamma(0) = (0, 0)$ and $|\gamma'(0)| > 0$. Put $g(t) = f(\gamma(t))$ and prove that g is differentiable for every $t \in R^1$.
If $\gamma \in \mathscr{C}'$, prove that $g \in \mathscr{C}'$.
(*d*) In spite of this, prove that f is not differentiable at $(0, 0)$.
Hint: Formula (40) fails.

(a) For $(x,y) \neq (0,0)$ we have

$$D_1 f(x,y) = \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2} \; ; \; D_2 f(x,y) = -\frac{2x^3 y}{(x^2+y^2)^2}$$

$$0 \leq D_1 f(x,y) = \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2} \leq \frac{x^2(3x^2+3y^2)}{(x^2+y^2)^2} = \frac{3x^2}{(x^2+y^2)} \leq \frac{3x^2}{x^2} = 3$$

$$|D_2 f(x,y)| = \left| \frac{x^2(2xy)}{(x^2+y^2)^2} \right| \leq \frac{x^2(x^2+y^2)}{(x^2+y^2)^2} = \frac{x^2}{x^2+y^2} \leq \frac{x^2}{x^2} = 1$$

For $(x,y) = (0,0)$,

$$D_1 f(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x \to 0} \frac{x-0}{x} = 1$$

$$D_2 f(0,0) = \lim_{x \to 0} \frac{f(0,x) - f(0,0)}{x} = \lim_{x \to 0} \frac{0}{x} = 0$$

Thus $D_1 f(0,0)$ & $D_2 f(0,0)$ bounded

(d) If f is differentiable at $(0,0)$ then

$$f(x,y) = f(0,0) + \left[x \frac{\partial f}{\partial x}(0,0) + y \frac{\partial f}{\partial y}(0,0)\right] + r(x,y)$$

where $\displaystyle\lim_{x,y \to (0,0)} \frac{r(x,y)}{\sqrt{x^2+y^2}} = 0$

we have $\frac{\partial f}{\partial x}(0,0) = D_1 f(0,0) = 1$, $\frac{\partial f}{\partial y}(0,0) = D_2 f(0,0) = 0$

then $f(x,y) = f(0,0) + x + r(x,y)$

$r(x,y) = f(x,y) - x = \dfrac{-xy^2}{x^2+y^2}$, so we must have $\displaystyle\lim_{(x,y)\to(0,0)} \frac{-xy^2}{(x^2+y^2)^{\frac{3}{2}}} = 0$

However when $x = y$, $\dfrac{-x^3}{(2x^2)^{\frac{3}{2}}} = -2^{-\frac{3}{2}} \neq 0$

Here comes contradiction, So f is not differentiable at $(0,0)$

Additional Problem 1: Let P be the set of polynomials on $[0,1]$ with the supremum norm

$$\|p\| = \sup_{x\in[0,1]} |p(x)|$$

And let $T : P \to P$ be defined by $T(p) = p'$

Show that there is no C such that $\|T(p)\| \leq C\,\|p\|$. This is an example of an unbounded linear transformation.

We prove this by assuming Contrary.

We assume that there is C s.t. $\|T(p)\| \leq C\|p\|$

That means for all polynomials p, we have

$$\boxed{\sup_{x\in[0,1]} |p'(x)| \leq C \cdot \sup_{x\in[0,1]} |p(x)| \qquad (*)}$$

In the following, we will construct a p, which does not satisfy $(*)$, leads to contradiction.

However take By Archimedean Property, there is $n \in \mathbb{N}$ s.t. $n > C$ and $n \geq 2$.

then take $p(x) = x^n$ we have

$p'(x) = n x^{n-1}$,

$n = \sup_{x\in[0,1]} |n x^{n-1}| > C = C\cdot 1 = C \cdot \sup |x^n|$, conflict with $(*)$

Thus we proved that there is no such C by constructing contradiction #

Additional Problem 2: Use the definition of a derivative learned in this course to give a new proof of the product rule. That is, if $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable at x, then so if fg and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

$f(x+h) = f(x) + hf'(x) + r_f(h)$ where $\lim_{h\to 0} \frac{r_f(h)}{h} = 0$

$g(x+h) = g(x) + hg'(x) + r_g(h)$ where $\lim_{h\to 0} \frac{r_g(h)}{h} = 0$

$$\begin{aligned}
(fg)(x+h) &= f(x+h)g(x+h) \\
&= \left(f(x) + hf'(x) + r_f(h)\right)\left(g(x) + hg'(x) + r_g(h)\right) \\
&= f(x)g(x) + hf(x)g'(x) + f(x)r_g(h) \\
&\quad + hf'(x)g(x) + hf'(x)hg'(x) + hf'(x)r_g(h) \\
&\quad + r_f(h)g(x) + r_f(h)hg'(x) + r_f(h)r_g(h) \\
&= f(x)g(x) + h\left(f(x)g'(x) + f'(x)g(x)\right) \\
&\quad + h^2 f'(x)g'(x) + f(x)r_g(h) + r_f(h)g(x) \\
&\quad + hf'(x)r_g(h) + hr_f(h)g'(h) + r_f(h)r_g(h)
\end{aligned}$$

$\underline{\text{denoted as}}$ $f(x)g(x) + h\left(f(x)g'(x) + f'(x)g(x)\right) + r_{fg}(h)$

$$\lim_{h\to 0} \frac{r_{fg}(h)}{h} = \lim_{h\to 0} hf'(x)g'(x) + \frac{f(x)r_g(h)}{h} + \frac{r_f(h)}{h}g(x) + f'(x)r_g(h) + r_f(h)g'(h) + \frac{r_f(h)r_g(h)}{h}$$

Because f, g differentiable at x, so $f'(x), g'(x)$ exist and bounded

So $\lim_{h\to 0} hf'(x)g'(x) = 0$;

Because f, g differentiable at x, so $f(x), g(x)$ bounded around x

So $\lim_{h\to 0} f(x)\frac{r_g(h)}{h} = \lim_{h\to 0} \frac{r_f(h)}{h}g(x) = 0$

Because $\lim_{h\to 0} h = 0$, $\lim_{h\to 0} \frac{r_f(h)}{h} = \lim_{h\to 0} \frac{r_g(h)}{h} = 0$ thus $\lim_{h\to 0} r_f(h) = 0 = \lim_{h\to 0} r_g(h)$

So $\lim_{h\to 0} f'(x)r_g(h) = \lim_{h\to 0} r_f(h)g'(x) = 0$

So $\lim_{h\to 0} \frac{r_f(h)}{h} r_g(h) = \lim_{h\to 0} \frac{r_f(h)}{h} \cdot \frac{r_g(h)}{h} \cdot h = 0$

Thus $\lim_{h\to 0} \frac{r_{fg}(h)}{h} = 0$, $fg(x+h) = f(x)g(x) + h\left(f'(x)g(x) + f(x)g'(x)\right) + r_{fg}(h)$ where $\lim_{h\to 0} \frac{r_{fg}(h)}{h} = 0$

Then by definition, we have $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ #