MATH S4062 – HOMEWORK 6

- **Chapter 9:** 6, 8, 14(a)(d)

Please also do the two Additional Problems below:

Additional Problem 1: Let P be the set of polynomials on $[0,1]$ with the supremum norm

$$\|p\| = \sup_{x\in[0,1]} |p(x)|$$

And let $T : P \to P$ be defined by $T(p) = p'$

Show that there is no C such that $\|T(p)\| \leq C\,\|p\|$. This is an example of an unbounded linear transformation.

Additional Problem 2: Use the definition of a derivative learned in this course to give a new proof of the product rule. That is, if $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable at x, then so if fg and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

Date: Due: Friday, July 29, 2022.

Hints:

Problem 6: For $(x, y) \neq (0, 0)$, you can just use the differentiation rules from calculus. For $(x, y) = (0, 0)$, you need to use the definition of $D_1 f = \frac{\partial f}{\partial x}$ and $D_2 f = \frac{\partial f}{\partial y}$ as limits. To show it's not continuous at $(0, 0)$, consider the path (x, x)

Problem 8: Here Rudin means that $f : \mathbb{R}^n \to \mathbb{R}$. For this consider $\phi(t) = f(x + ty)$ and you can use the usual rules from Calculus.

Problem 14: Just do (a) and (d) here. For (a), calculate $\frac{\partial f}{\partial x_1}$ both for $(x, y) \neq (0, 0)$ (differentiation rules) and for $(x, y) = 0$ (limit definition) and same with x_2. For (d), if f were differentiable at $(0, 0)$ then we would have

$$f(x, y) = f(0, 0) + \frac{\partial f}{\partial x}(0, 0)x + \frac{\partial f}{\partial y}(0, 0)y + r(x, y)$$

With $\lim_{(x,y)\to(0,0)} \frac{|r(x,y)|}{\sqrt{x^2+y^2}} = 0$

Use the equation above to solve for $r(x, y)$ and show that the limit is not 0 by considering the path (x, x)

Additional Problem 1: All you need to do is to find a sequence of (easy) polynomials $p_n \in P$ such that $\lim_{n\to\infty} \frac{\|T(p_n)\|}{\|p_n\|} = \infty$

Additional Problem 2: Start with $(fg)(x + h) = f(x + h)g(x + h)$ and write this in terms of $f(x)g(x)$.

Note: For a solution to this, check out Cool Product Rule proof