MATH 409 – HOMEWORK 7

Reading: Sections 15 and 17. In section 15, ignore the proof of the alternating series test. There will be more section 17 problems next time.

- **Section 15:** 3, 6 (see Note), AP1, AP2, AP3, AP4, AP5, AP6 (optional: AP8)
- **Section 17:** 9, 10(a)(b) (see Note), AP7

Note: For Problem 6(b), please show the result directly, without using Exercise 14.7

Note: For Problem 10(a), please do this using both the sequence definition *and* the $\epsilon - \delta$ definition.

Additional Problem 1: Prove the Integral Test:

Integral Test:

If $f(x) \geq 0$ is decreasing on $[1, \infty)$, then:

(1) $\int_1^\infty f(x)dx = \infty \Rightarrow \sum_{n=1}^\infty f(n) = \infty$

(2) $\int_1^\infty f(x)dx < \infty \Rightarrow \sum_{n=1}^\infty f(n)$ converges

Additional Problem 2: There is an important inequality for series called the Cauchy-Schwarz inequality.

Date: Due: Friday, October 22, 2021.

Cauchy-Schwarz Inequality:

$$\left|\sum_{n=1}^{\infty} a_n b_n\right| \leq \left(\sum_{n=1}^{\infty} (a_n)^2\right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} (b_n)^2\right)^{\frac{1}{2}}$$

Use the Cauchy-Schwarz inequality to prove the following:

(a) If $a_n \geq 0$ and $\sum a_n$ converges, then the following series converges

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$$

(b) If a_n and $b_n \geq 0$ both $\sum a_n$ and $\sum b_n$ converge, then the following series converges

$$\sum_{n=1}^{\infty} \sqrt{a_n b_n}$$

Note: The following video covers part (a): bprp vs. Dr Peyam Battle

Aside: If you want to see a slick proof of the Cauchy-Schwarz inequality (in its general form), check out: Cauchy-Schwarz Proof

Additional Problem 3: Let (s_n) be the sequence defined by

$$s_n = \left(\sum_{k=1}^{n} \frac{1}{k}\right) - \ln(n) = \left(\sum_{k=1}^{n} \frac{1}{k}\right) - \int_1^n \frac{1}{x} dx$$

(a) Show that (s_n) is decreasing

(b) Show that $0 \leq s_n \leq 1$ for all n

(c) Conclude that (s_n) converges.

Cultural Note: The limit of (s_n) is denoted by γ and is called the Euler-Mascheroni constant. Even though $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ and $\ln(n) \to \infty$, this problem shows that the difference between the two is actually finite! It is not known if γ is rational or not.

Additional Problem 4: There is another comparison test used frequently in Calculus:

Limit Comparison Test:

If $a_n, b_n \geq 0$ and $b_n \neq 0$ for all n and

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c$$

with $0 < c < \infty$, then either both $\sum a_n$ and $\sum b_n$ converge, or both diverge

(a) Apply the limit comparison test to determine if the following series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n^3 - 1}{n^4 + 3}$$

(b) Prove the limit comparison test

Additional Problem 5: Recall the definition of e from last time:

Definition:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Show that e is irrational (see hints)

Additional Problem 6: Calculate the sum of the following series (see hints)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Definition:

$f : \mathbb{R} \to \mathbb{R}$ is **Lipschitz** if there is a constant $C > 0$ such that for all a and b, we have

$$|f(b) - f(a)| \leq C\,|b - a|$$

Additional Problem 7: Show that if f is Lipschitz then f is continuous

Optional Additional Problem 8:

(a) Prove the following generalization of the Limit Comparison Test (from AP4):

Limsup Comparison Test:

If $a_n, b_n \geq 0$ and $b_n \neq 0$ for all n and

$$\limsup_{n\to\infty} \frac{a_n}{b_n} = c$$

with $0 \leq c < \infty$, then if $\sum b_n$ converges, then $\sum a_n$ converges.

(b) Use (a) to figure out if the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{(-1)^n + 1}{n^2}$$

Observe that you cannot apply the regular limit comparison test to this!

Hints:

15.6(b): Use the fact that for large n, $a_n < 1$ (why?) and therefore $(a_n)^2 \leq a_n$. It's best to use the Cauchy criterion in my opinion.

17.9(a)(d) Do the usual trick of assuming $|x - x_0| < 1$ and therefore

$$|x| = |x - x_0 + x_0| \leq |x - x_0| + |x_0| = 1 + |x_0|$$

Yes, the constant gets unusually big for (d) ☺

17.10(a) I did a very similar problem in the following video: Not continuous

AP1: It's *literally* the same proof as the one in lecture (or in the book), except you replace $\frac{1}{x}$ by $f(x)$ and $\frac{1}{n}$ by $f(n)$. The only other minor replacement is that, in the proof of convergence, instead of doing $1+$ Rectangles 2 to n , you do $f(1)+$ Rectangles 2 to n

AP3: For (a), show $s_{n+1} - s_n < 0$; a picture might be helpful here. For (b), since (s_n) is decreasing, $s_n \leq s_1$. Moreover, notice that in the proof of the integral test, we showed something stronger, namely that $\sum_{k=1}^{n} \frac{1}{k} \geq \int_1^{n+1} \frac{1}{x} dx$

AP4(b): Since $c > 0$, let $\epsilon > 0$ be such that $\epsilon < c$, and then use the definition of the limit to show that $(c - \epsilon) b_n \leq a_n \leq (c + \epsilon) b_n$ and then use the usual comparison test.

AP5: First, with $s_n = \sum_{k=0}^{n} \frac{1}{k!}$, show that:

$$\begin{aligned} e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \dots \\ &< \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots\right) \text{ (Geometric series)} \\ &= \frac{1}{n!n} \end{aligned}$$

Hence $0 < e - s_n < \frac{1}{(n!)n}$

Now if e were rational, then $e = \frac{p}{q}$ where $p, q > 0$ are integers.

Then show that $(q!)\,e$ is an integer and that

$$(q!)\,s_q = q!\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!}\right)$$

is an integer, and conclude $q!(e - s_q)$ is an integer.

On the other hand, show using the above with $n = q$ that

$$0 < (q!)\,(e - s_q) < \frac{1}{q}$$

And find a (juicy) contradiction.

Note: You can find solutions in this video: e is irrational

AP 6: Use the following formula with $f(x) = x$

Parseval's Identity:

$$\text{If} \quad A_n = \left(\frac{2}{\pi}\right) \int_0^{\pi} f(x) \sin(nx) dx$$

$$\text{Then} \quad \sum_{n=1}^{\infty} (A_n)^2 = \left(\frac{2}{\pi}\right) \int_0^{\pi} (f(x))^2 \, dx$$

For this, first calculate A_n using integration by parts, and then plug your formula for A_n in the sum above.

Note: You can find solutions in this video: Sum of $\frac{1}{n^2}$