HOMEWORK 7 – AP SOLUTIONS

AP 1

(a) Consider the partial sums:

$$s_n = \sum_{k=1}^{n} f(k) = f(1) + \cdots + f(n)$$

As before, for each $k = 1, \ldots, n$ consider the rectangle with base $[k, k+1]$ and height $f(k)$

Then

$$s_n = f(1) + \cdots + f(n) = \text{ Sum of areas of } n \text{ rectangles}$$

On the other hand, since f is decreasing, the above sum *larger* than the area under f from 1 to $n+1$ that is $\int_1^{n+1} f(x)dx$.

And therefore

$$s_n = \sum_{k=1}^{n} f(k) \geq \int_1^{n+1} f(x)dx =: t_n$$

However

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} \int_1^{n+1} f(x)dx = \int_1^{\infty} f(x)dx = \infty \quad \text{(By assumption)}$$

Date: Friday, October 22, 2021.

And therefore, by comparison, $\lim_{n\to\infty} s_n = \infty$, meaning that $\sum_{n=1}^{\infty} f(n) = \infty$ (by definition of a series) $\square$

(b) Consider again the partial sums

$$s_n = \sum_{k=1}^{n} f(k) = f(1) + \cdots + f(n)$$

It is enough to show that (s_n) is bounded.

This time, for each $k = 1, \ldots, n$, consider the rectangle with base $[k-1, k]$ and height $f(k)$

$$s_n = f(1) + \cdots + f(n) = \text{ Sum of the areas of the rectangles}$$

Note: Since f and $\int_1^\infty f(x)dx$ is only defined on $[1, \infty)$, we need to ignore the first rectangle (which has finite area anyway), so

$$s_n = \text{ (Rectangle 1) + (Rectangles 2 to n) } = f(1) + \text{ (Rectangles 2 to n)}$$

Since f is decreasing, the area under the graph of f from 1 to n is **bigger** than the sum of the areas of rectangles 2 to n

$$\begin{aligned} s_n &\leq f(1) + \text{ Area of Rectangles 2 to n} \\ &\leq f(1) + \int_1^n f(x)dx \\ &\leq f(1) + \int_1^\infty f(x)dx \text{ (since } f \geq 0) \end{aligned}$$

Therefore, with $M =: f(1) + \int_1^\infty f(x)dx$ we get $0 \leq s_n \leq M$

Hence $|s_n| \leq M$ for all n, and so (s_n) is bounded, and therefore $\sum f(n)$ converges $\square$

AP 2

(a) By Cauchy-Schwarz, we get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\sqrt{a_n}}{n} &= \sum_{n=1}^{\infty}\sqrt{a_n}\left(\frac{1}{n}\right)\\ &\leq \left(\sum_{n=1}^{\infty}(\sqrt{a_n})^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^2\right)^{\frac{1}{2}}\\ &\leq \left(\sum_{n=1}^{\infty}a_n\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}}\end{aligned}$$

However $\sum_{n=1}^{\infty} a_n < \infty$ by assumption, and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ since it's a 2−series, so the right-hand-side is finite, and therefore $\sum_{n=1}^{\infty}\frac{\sqrt{a_n}}{n}$ is bounded, and hence converges.

(b) Again, by the Cauchy-Schwarz inequality

$$\begin{aligned}\sum_{n=1}^{\infty}\sqrt{a_n}\sqrt{b_n} &\leq \left(\sum_{n=1}^{\infty}(\sqrt{a_n})^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}\left(\sqrt{b_n}\right)^2\right)^{\frac{1}{2}}\\ &\leq \left(\sum_{n=1}^{\infty}a_n\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}b_n\right)^{\frac{1}{2}}\end{aligned}$$

But by assumption, each term on the right-hand-side is finite, and therefore $\sum_{n=1}^{\infty}\sqrt{a_n b_n}$ is bounded, and hence converges.

AP 3

(a)

$$\begin{aligned} s_{n+1} - s_n &= \left(\sum_{k=1}^{n+1} \frac{1}{k}\right) - \int_1^{n+1} \frac{1}{x} dx - \left(\sum_{k=1}^{n} \frac{1}{k}\right) - \int_1^{n} \frac{1}{x} dx \\ &= \left(\sum_{k=1}^{n+1} \frac{1}{k}\right) - \left(\sum_{k=1}^{n} \frac{1}{k}\right) - \left(\int_1^{n+1} \frac{1}{x} dx - \int_1^{n} \frac{1}{x} dx\right) \\ &= \frac{1}{n+1} - \int_n^{n+1} \frac{1}{x} dx \end{aligned}$$

However, since $\frac{1}{x}$ is decreasing, we have $\frac{1}{x} \geq \frac{1}{n+1}$ on the interval $[n, n+1]$, hence the area under f on $[n, n+1]$, which is $\int_n^{n+1} \frac{1}{x} dx$ is greater than the area of the rectangle with base $[n, n+1]$ and height $\frac{1}{n+1}$, and so

$$s_{n+1} - s_n = \frac{1}{n+1} - \int_n^{n+1} \frac{1}{x} dx < 0$$

Hence $s_{n+1} < s_n$ and therefore (s_n) is decreasing

(b) First of all, since (s_n) is decreasing, we have

$$s_n \leq s_1 = \left(\sum_{k=1}^{1} \frac{1}{n}\right) - \int_1^1 \frac{1}{x} dx = 1 - 0 = 1$$

Hence $s_n \leq 1$.

On the other hand, by considering again the rectangles with base $[k, k+1]$ and height $\frac{1}{k}$ (for $k = 1, \ldots, n$), we get that

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{k} &= \text{Sum of areas of rectangles}\\ &\geq \text{Area of } \frac{1}{x} \text{ from 0 to } n+1\\ &= \int_1^{n+1}\frac{1}{x}dx\\ &> \int_1^{n}\frac{1}{x}dx\end{aligned}$$

And therefore $s_n = \sum_{k=1}^{n}\frac{1}{k} - \int_1^n \frac{1}{x}dx > 0$.

Hence we conclude that $0 < s_n \leq 1$ for all n.

(c) Since (s_n) is decreasing and bounded below by 0, (s_n) converges.

AP 4

(a) Let $a_n = \frac{n^3-1}{n^4+3}$ and $b_n = \frac{n^3}{n^4} = \frac{1}{n}$. Then

$$\frac{a_n}{b_n} = \frac{\frac{n^3-1}{n^4+3}}{\frac{1}{n}} = \frac{n(n^3-1)}{n^4+3} = \frac{n^4-n}{n^4+3} \overset{n\to\infty}{\to} 1$$

But since $\sum b_n = \sum \frac{1}{n} = \infty$, by the limit comparison test, we conclude that $\sum a_n$ diverges as well.

(b) Since $c > 0$, let $\epsilon > 0$ be such that $c - \epsilon > 0$, then by definition of a limit, there is N such that if $n > N$, then

$$\begin{aligned} \left|\frac{a_n}{b_n} - c\right| < \epsilon \Rightarrow & -\epsilon < \frac{a_n}{b_n} - c < \epsilon \\ \Rightarrow & c - \epsilon < \frac{a_n}{b_n} < c + \epsilon \\ \Rightarrow & (c-\epsilon) b_n < a_n < (c+\epsilon) b_n \end{aligned}$$

However, if $\sum b_n$ converges, then since $0 \leq a_n < (c+\epsilon)b_n$, by comparison $\sum a_n$ converges

And if $\sum b_n = \infty$, then since $a_n > (c-\epsilon)b_n$, we get that by comparison $\sum a_n = \infty$ $\square$

AP 5

STEP 1: First of all:

$$
\begin{aligned}
e - s_n &= \sum_{k=0}^{\infty} \frac{1}{k!} - \sum_{k=0}^{n} \frac{1}{k!} \\
&= \sum_{k=n+1}^{\infty} \frac{1}{k!} \\
&= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \dots \\
&= \frac{1}{(n+1)!} \left(1 + \frac{(n+1)!}{(n+2)!} + \frac{(n+1)!}{(n+3)!} + \dots \right) \\
&= \frac{1}{(n+1)!} \left(1 + \frac{1}{n+2} + \frac{1}{(n+3)(n+2)} + \dots \right) \\
&< \frac{1}{(n+1)!} \left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots \right) \\
&= \frac{1}{(n+1)!} \left(1 + \left(\frac{1}{n+1} \right) + \left(\frac{1}{n+1} \right)^2 \right) \\
&= \frac{1}{(n+1)!} \left(\frac{1}{1 - \left(\frac{1}{n+1}\right)} \right) \\
&= \frac{1}{(n+1)!} \left(\frac{1}{\frac{n+1-1}{n+1}} \right) \\
&= \frac{1}{(n+1)!} \left(\frac{n+1}{n} \right) \\
&= \frac{n+1}{(n+1)!} \left(\frac{1}{n} \right) \\
&= \frac{1}{n!} \left(\frac{1}{n} \right)
\end{aligned}
$$

Therefore $e - s_n < \frac{1}{n!}\left(\frac{1}{n}\right)$ but also $e - s_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} > 0$ and therefore

$$0 < e - s_n < \frac{1}{n!n}$$

STEP 2: Now suppose $e = \frac{p}{q}$, where $p, q > 0$ since $e > 0$. But then

$$q!e = q!\left(\frac{p}{q}\right) = \left(\frac{q!}{q}\right)p = (q-1)(q-2)\dots(1)p, \text{ which is an integer}$$

And also

$$\begin{aligned} q!s_q =& q!\left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!}\right) \\ =& q! + q! + \frac{q!}{2!} + \cdots + \frac{q!}{q!} \\ =& q! + q! + q(q-1)\dots(3) + \cdots + 1 \text{ is an integer} \end{aligned}$$

Therefore $q!(e - s_q) = q!e - q!s_q$ is an integer, being the difference of two integers.

STEP 3: However, in **STEP 1** with $n = q \geq 1$, we get that

$$0 < e - s_q < \frac{1}{q!q} \Rightarrow 0 < q!(e - s_q) < \frac{1}{q} \leq 1 \Rightarrow 0 < q!(e - s_q) < 1$$

But then $q!(e - s_q)$ is an integer between 0 and 1, which is impossible $\Rightarrow\Leftarrow$

AP 6

On the one hand, using an integration by parts, we get

$$\begin{aligned} A_n =& \frac{2}{\pi}\left(\int_0^\pi x\sin(nx)dx\right) \\ =& \frac{2}{\pi}\left(\left[x\left(\frac{-\cos(nx)}{n}\right)\right]_0^\pi + \int_0^\pi \frac{\cos(nx)}{n}dx\right) \\ =& \frac{2}{\pi}\left(\frac{-\pi\cos(\pi n)}{n} + 0 + \left[\frac{\sin(nx)}{n^2}\right]_0^\pi\right) \\ =& \frac{2}{\pi}\left[\left(-\pi(-1)^n\right)n + 0 - 0\right] \\ =& \frac{-2(-1)^n}{n} \end{aligned}$$

On the other hand:

$$\frac{2}{\pi}\int_0^\pi x^2dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^\pi = \left(\frac{2}{\pi}\right)\left(\frac{\pi^3}{3}\right) = \frac{2\pi^2}{3}$$

Therefore Parseval's Identity becomes:

$$\begin{aligned} \sum_{n=1}^{\infty}(A_n)^2 =& \left(\frac{2}{\pi}\right)\int_0^\pi x^2dx \\ \sum_{n=1}^{\infty}\left(\frac{-2(-1)^n}{n}\right)^2 =& \frac{2\pi^2}{3} \\ \sum_{n=1}^{\infty}\frac{4}{n^2} =& \frac{2\pi^2}{3} \\ \sum_{n=1}^{\infty}\frac{1}{n^2} =& \frac{2\pi^2}{4(3)} \\ \sum_{n=1}^{\infty}\frac{1}{n^2} =& \frac{\pi^2}{6} \end{aligned}$$

AP 7

Let $\epsilon > 0$ be given, let $\delta = \frac{\epsilon}{C}$, then if $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| \leq C\,|x - x_0| < C\left(\frac{\epsilon}{C}\right) = \epsilon \checkmark$$

Hence f is continuous at x_0, and hence continuous $\square$

AP 8

(a) Let $\epsilon > 0$ be given, by assumption we have

$$\limsup_{n=\infty} \frac{a_n}{b_n} = \lim_{N\to\infty} \sup\left\{\frac{a_n}{b_n} \mid n > N\right\} = c$$

So there is N_1 such that if $N > N_1$, then

$$\left|\sup\left\{\frac{a_n}{b_n} \mid n > N\right\} - c\right| < \epsilon \Rightarrow \sup\left\{\frac{a_n}{b_n} \mid n > N\right\} - c < \epsilon$$

In particular, for some N we have

$$\sup\left\{\frac{a_n}{b_n} \mid n > N\right\} < c + \epsilon$$

And so for all $n > N$, we get

$$\frac{a_n}{b_n} < c + \epsilon \Rightarrow a_n < (c + \epsilon) b_n$$

However, since $\sum b_n < \infty$, by comparison, we get $\sum a_n < \infty$, and so $\sum a_n$ converges $\square$

(b) Let $a_n = \frac{(-1)^n+1}{n^2}$ and $b_n = \frac{1}{n^2}$, then

$$\limsup_{n\to\infty} \frac{a_n}{b_n} = \limsup_{n\to\infty} \frac{\frac{(-1)^n+1}{n^2}}{\frac{1}{n^2}} = \limsup_{n\to\infty} (-1)^n + 1 = 2 < \infty$$

Therefore, since $\sum b_n = \sum \frac{1}{n^2}$ converges, we get $\sum a_n = \sum \frac{(-1)^n+1}{n^2}$ converges