HOMEWORK 7 – SELECTED BOOK SOLUTIONS

17.9(a)

STEP 1: Scratchwork

$$\begin{aligned}|f(x) - f(x_0)| &= |f(x) - f(2)| \\ &= \left|x^2 - 4\right| \\ &= |x-2|\,|x+2| \\ &\stackrel{?}{<} \epsilon\end{aligned}$$

Now if $|x-2| < 1$, then $-1 < x-2 < 1$ so $1 < x < 3$, hence $3 < x+2 < 5$ and therefore $|x+2| < 5$. Hence

$$|x-2|\,|x+2| \leq 5\,|x-2| < \epsilon \Rightarrow |x-2| < \frac{\epsilon}{5}$$

STEP 2: Actual proof

Let $\epsilon > 0$ be given, let $\delta = \min\left\{1, \frac{\epsilon}{5}\right\} > 0$ and suppose $|x-2| < \delta$, then $|x+2| < 5$ and therefore

$$|f(x) - f(2)| = |x-2|\,|x+2| \leq 5\,|x-2| < 5\left(\frac{\epsilon}{5}\right) = \epsilon\checkmark$$

Hence $f(x) = x^2$ is continuous at $x_0 = 2$ □

Date: Friday, October 22, 2021.

17.9(b)

STEP 1: Scratchwork

$$|f(x) - f(x_0)| = \left|\sqrt{x}\right| = \sqrt{x} < \epsilon \Rightarrow |x| < \epsilon^2$$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, let $\delta = \epsilon^2$, then if $|x| < \delta$, then

$$|f(x) - f(x_0)| = \left|\sqrt{x}\right| < \sqrt{\epsilon^2} = \epsilon\checkmark$$

Hence $f(x) = \sqrt{x}$ is continuous at $x_0 = 0$ $\square$

17.9(c)

STEP 1: Scratchwork

$$|f(x) - f(x_0)| = \left|x \sin\left(\frac{1}{x}\right)\right| = |x| \underbrace{\left|\sin\left(\frac{1}{x}\right)\right|}_{\leq 1} \leq |x| < \epsilon$$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, let $\delta = \epsilon$, then if $|x| < \delta$, then

$$|f(x) - f(x_0)| = |x| \underbrace{\left|\sin\left(\frac{1}{x}\right)\right|}_{\leq 1} \leq |x| < \epsilon\checkmark$$

Hence $f(x)$ is continuous at $x_0 = 0$ $\square$

17.9(D)

STEP 1: Scratchwork

$$\begin{aligned} |f(x) - f(x_0)| &= \left|x^3 - (x_0)^3\right| \\ &= |x - x_0| \left|x^2 + x_0 x + (x_0)^2\right| \\ &\leq |x - x_0| \left(|x|^2 + |x_0|\,|x| + |x_0|^2\right) \\ &\overset{?}{<} \epsilon \end{aligned}$$

Now if $|x - x_0| < 1$, then

$$|x| = |x - x_0 + x_0| \leq |x - x_0| + |x_0| < 1 + |x_0|$$

And therefore

$$\begin{aligned} &|x - x_0| \left(|x|^2 + |x_0|\,|x| + |x_0|^2\right) \\ \leq &|x - x_0| \underbrace{\left((1 + |x_0|)^2 + |x_0|\,(1 + |x_0|) + |x_0|^2\right)}_{C} \\ < &\epsilon \end{aligned}$$

Which gives $\delta = \frac{\epsilon}{C}$

STEP 2: Actual proof

Let $\epsilon > 0$ be given, let $\delta = \min\left\{1, \frac{\epsilon}{C}\right\} > 0$ and suppose $|x - x_0| < \delta$, then $|x| \leq |x_0| + 1$ and therefore

$$
\begin{aligned}
|f(x) - f(x_0)| &= |x - x_0| \left| x^2 + x_0 x + (x_0)^2 \right| \\
&\leq |x - x_0| \left(|x|^2 + |x_0| \, |x| + |x_0|^2 \right) \\
&\leq |x - x_0| \, C \\
&< \left(\frac{\epsilon}{\not{C}} \right) \not{C} \\
&= \epsilon \checkmark
\end{aligned}
$$

Hence $f(x) = x^3$ is continuous at x_0 □

17.10(a)

The Sequence Way: Let (x_n) be a sequence of positive numbers converging to 0, for instance $x_n = \frac{1}{n}$. Then $x_n \to 0$, but

$$f(x_n) = 1 \nrightarrow 0 = f(0)$$

Therefore f is not continuous at $x_0 = 0$.

The $\epsilon - \delta$ Way: Let $\epsilon = \frac{1}{2}$, then, if $\delta > 0$, let $x =$ any positive number such that $|x| < \delta$, for instance $x = \frac{\delta}{2}$, then $|x - 0| < \delta$, but

$$|f(x) - f(0)| = |1 - 0| = 1 \not< \frac{1}{2} = \epsilon$$

Hence f is not continuous at $x_0 = 0$

17.10(b)

Choose x_n such that $\sin\left(\frac{1}{x_n}\right) = 1$, so $\frac{1}{x_n} = \frac{\pi}{2} + 2\pi n$, so

$$x_n = \frac{1}{\frac{\pi}{2} + 2\pi n}$$

Then $x_n \to 0$ but $g(x_n) = \sin(\frac{\pi}{2} + 2\pi n) = 1 \nrightarrow g(0) = 0$.

Hence g is not continuous at $x_0 = 0$