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MATH S4062 – HOMEWORK 7

- **Chapter 9:** 16, 17(b)(c), 19, 23

Please also do the Additional Problem below:

Additional Problem 1: Show that the Implicit Function Theorem implies (the following version of) the Inverse Function Theorem:

Inverse Function Theorem: Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. If $\det(f'(a)) \neq 0$ for some a, there is an open neighborhood U of a and an open neighborhood V of $f(a)$ and $g : V \to U$ such that $f(g(y)) = y$ for all y. Moreover

$$g'(f(a)) = (f'(a))^{-1}$$

Note: Hence, combined with the proof given in lecture, the Inverse and Implicit function theorems are equivalent.

Date: Due: Tuesday, August 2, 2022.

Hints:

Problem 16: For the last part, it's useful to notice that $f'\left(\frac{1}{\pi m}\right) = 1 + 2(-1)^m$ so the minimum value of f inside $\left[\frac{1}{(2m+1)\pi}, \frac{1}{(2m)\pi}\right]$ is assumed at an interior point.

Problem 17: The Jacobian is just the determinant of the derivative matrix and f is not one-to-one by periodicity in y. For (c), it's useful to let $u = e^x \cos(y)$ and $v = e^x \sin(y)$ and solve for x and y in terms of u and v

Problem 19: **Please use the Implicit Function Theorem for the first three parts.** For the last part, if you add the last two equations and subtract it from the first, you should get $u = 0$ or $u = 3$, so generally the system cannot be solved for x, y, z in terms of u.

Problem 23: **Please do this using the Implicit Function Theorem**

Additional Problem 1: Apply the Implicit Function theorem to $F : \mathbb{R}^{n+n} \to \mathbb{R}^n$ given by $F(x, y) = y - f(x)$ and $(x_0, y_0) = (a, f(a))$ and let $g(y) = G(y)$, where G is from the Implicit Function Theorem. Careful that here the roles of x and y are switched

16. Show that the continuity of $\mathbf{f}'$ at the point $\mathbf{a}$ is needed in the inverse function theorem, even in the case $n = 1$: If

$$f(t) = t + 2t^2 \sin\left(\frac{1}{t}\right)$$

for $t \neq 0$, and $f(0) = 0$, then $f'(0) = 1$, f' is bounded in $(-1, 1)$, but f is not one-to-one in any neighborhood of 0.

$f(0)=0 \quad f(0) = 0 + 2 \cdot 0^2 \sin\frac{1}{t} = 0$

$f'(0) = \lim_{t\to 0} \dfrac{f(t)}{t} = \lim_{t\to 0} \left(1 + 2t \cdot \sin\frac{1}{t}\right) = 1$

$f'(t) = 1 + 4t\sin\frac{1}{t} - 2\cos\left(\frac{1}{t}\right)$ for $t \neq 0$ $\quad (|f'(t)| \leq 1+4+2=7)$

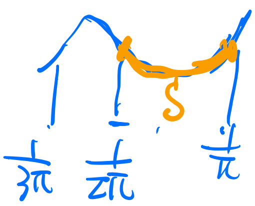
$\Rightarrow |f'(t)| \leq 7$ for all $t \in (-1,1)$

$f'\left(\frac{1}{\pi m}\right) = 1 + 4 \cdot 0 - 2 \cdot \cos(\pi m)$

$\quad = 1 - 2(-1)^m$

so that $f(t)$ is decreasing at $t = \frac{1}{m\pi}$ if m is even $\left(f'\left(\frac{1}{\pi m}\right) < 0\right)$

and increase if k is odd



the minimum value of $f(t)$ on

$\left[\frac{1}{2m\pi}, \frac{1}{(2m-1)\pi}\right]$ is assumed

as an interior point s:

there is $s \in \left(\frac{1}{2m\pi}, \frac{1}{(2m-1)\pi}\right)$ such that, for all $t \in \left[\frac{1}{(2m\pi)}, \frac{1}{(2m-1)\pi}\right]$

$f(s) \leq f(t)$

Because f is continuous, $f(t)$ cannot be one-to-one on this interval.

17. Let $\mathbf{f} = (f_1, f_2)$ be the mapping of R^2 into R^2 given by

$$f_1(x, y) = e^x \cos y, \qquad f_2(x, y) = e^x \sin y.$$

(*a*) What is the range of f?

(*b*) Show that the Jacobian of f is not zero at any point of R^2. Thus every point of R^2 has a neighborhood in which f is one-to-one. Nevertheless, f is not one-to-one on R^2.

(*c*) Put $\mathbf{a} = (0, \pi/3)$, $\mathbf{b} = f(\mathbf{a})$, let $\mathbf{g}$ be the continuous inverse of $\mathbf{f}$, defined in a neighborhood of $\mathbf{b}$, such that $\mathbf{g}(\mathbf{b}) = \mathbf{a}$. Find an explicit formula for $\mathbf{g}$, compute $\mathbf{f}'(\mathbf{a})$ and $\mathbf{g}'(\mathbf{b})$, and verify the formula (52).

(*d*) What are the images under $\mathbf{f}$ of lines parallel to the coordinate axes?

(b) $$\begin{vmatrix} \frac{\partial f_1}{\partial x}, & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x}, & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x\cos y & -e^x\sin y \\ e^x \sin y & e^x\cos y \end{vmatrix}$$

$|f'(x)| = e^{2x}(\cos^2 y + \sin^2 y) = e^{2x} \neq 0$ for all x

Thus by inverse function Theorem, every point of $\mathbb{R}^2$ has a neighborhood in which f is one to one

However $f(x, y+2\pi) = f(x, y)$.

f is not one-to-one

(C) $a = (0, \frac{\pi}{3})$, $b = (\frac{1}{2}, \frac{\sqrt{3}}{2})$

$f_1(x, y) = e^x \cos y, \qquad f_2(x, y) = e^x \sin y.$

(52) $\mathbf{g}'(\mathbf{y}) = \{\mathbf{f}'(\mathbf{g}(\mathbf{y}))\}^{-1} \qquad (\mathbf{y} \in V).$

$u = e^x\cos y$, $v = e^x \sin y$

$u^2+v^2 = e^{2x} \Rightarrow x = \ln\sqrt{u^2+v^2}$

$\frac{v}{u} = \tan y \Rightarrow y = \arctan(\frac{v}{u})$

$\Rightarrow g(u,v) = (\ln\sqrt{u^2+v^2}, \arctan\frac{v}{u})$

$$f'(x,y) = \begin{pmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{pmatrix} \qquad g'(u,v) = \begin{pmatrix} \frac{u}{u^2+v^2}, & \frac{v}{u^2+v^2} \\ \frac{-v}{u^2+v^2}, & \frac{u}{u^2+v^2} \end{pmatrix}$$

$$g'(f(x,y)) = \begin{pmatrix} e^{-x}\cos y & e^{-x}\sin y \\ -e^{-x}\sin y & e^{-x}\cos y \end{pmatrix} \Rightarrow g'(f(x,y))f'(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f'(g(u,v)) = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \Rightarrow f'(g(x,y))g'(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which verify (52) #

19. Show that the system of equations

$$3x + y - z + u^2 = 0 \quad ①$$
$$x - y + 2z + u = 0 \quad ②$$
$$2x + 2y - 3z + 2u = 0 \quad ③$$

can be solved for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x; but not for x, y, z in terms of u.

③ + ② − ① we have $3u - u^2 = 0$

$\Rightarrow u = 3$ or 0

If one of two equations holds, we can solve just the last two equations for any two of x, y, z in terms of the third. The remaining equation will then satisfied.

1> $x = -\frac{z}{4}$, $y = \frac{7z}{4}$, $u = 0$; $x = -\frac{9+z}{4}$ $y = \frac{3+7z}{4}$, $u = 3$

2> $x = -\frac{y}{7}$, $z = \frac{4y}{7}$, $u = 0$; $x = \frac{60+4y}{7}$, $z = \frac{4y-3}{7}$, $u = 3$

3> $y = -7x$, $z = -4x$, $u = 0$; $y = \frac{7x-60}{4}$, $z = 9 - 4x$, $u = 3$

We can also see this through lineor transformation:

$$f'(x, y, z, u) = \begin{pmatrix} 3 & 1 & -1 & 2u \\ 1 & -1 & 2 & 1 \\ 2 & 1 & -3 & 2 \end{pmatrix}$$

Any 3×3 sub matrix containing the last column is invertible when $u = 0$ or $u = 3$

while the first three colum does not create an invertible matrix

23. Define f in R^3 by

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2 .$$

Show that $f(0, 1, -1) = 0$, $(D_1 f)(0, 1, -1) \neq 0$, and that there exists therefore a differentiable function g in some neighborhood of $(1, -1)$ in R^2, such that $g(1, -1) = 0$ and

$$f(g(y_1, y_2), y_1, y_2) = 0.$$

Find $(D_1 g)(1, -1)$ and $(D_2 g)(1, -1)$.

Implicit Function Theorem:

Suppose $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is C^1 and $F(x_0, y_0) = 0$ for some (x_0, y_0).

If $\det F_y(x_0, y_0) \neq 0$ then there is an open neighborhood U of (x_0, y_0) and an open neighborhood W of x_0 and a function $G : W \to \mathbb{R}^m$ differentiable at x_0 such that

$$\{(x, y) \in U \mid F(x, y) = 0\} = \{(x, G(x)) \mid x \in W\}$$

$$\text{Moreover } G'(x_0) = -\left(F_y(x_0, y_y)\right)^{-1} F_x(x_0, y_0)$$

In other words, if the derivative with respect to the variable you want to solve for is invertible, then the equation $F(x, y) = 0$ is locally the graph of a function $y = G(x)$.

$$f(0,1,-1) = 0 + e^0 + -1 = 1 - 1 = 0$$

$$(D_1 f)(x, y_1, y_2) = 2xy_1 + e^x$$

$$(D_1 f)(0, 1, -1) = 1 \neq 0$$

We can use the chain Rule

let $\varphi(y_1, y_2) = f(g(y_1, y_2), y_1, y_2) \equiv 0$

$$\frac{\partial \varphi}{\partial y_1}(y_1, y_2) = \frac{\partial f}{\partial x}(g(y_1, y_2), y_1, y_2)\frac{\partial g}{\partial y_1}(y_1, y_2) + \frac{\partial f}{\partial y_1} f(g, y_1, y_2)$$

$$\Rightarrow 0 = \left(2y_1 g(y_1, y_2) + e^{g(y_1, y_2)}\right)\frac{\partial g}{\partial y_1}(y_1, y_2) + \left(g(y_1, y_2)\right)^2$$

taking $y_1 = 1$, $y_2 = -1$, $g(y_1, y_2) = 0$, we get

$$0 = 1 \cdot D_1 g(1, -1) + 0 \Rightarrow D_1 g(1, -1) = 0$$

$$\frac{\partial \varphi}{\partial y_2} = \frac{\partial f}{\partial x}(g(y_1, y_2), y_1, y_2)\frac{\partial g}{\partial y_2}(y_1, y_2) + \frac{\partial f}{\partial y_2} f(g, y_1, y_2)$$

$$\Rightarrow 0 = \left(2y_1 g(y_1, y_2) + e^{g(y_1, y_2)}\right)\frac{\partial g}{\partial y_2}(y_1, y_2) + 1$$

taking $y_1 = 1$, $y_2 = -1$, $g(y_1, y_2) = 0$, we get

$$0 = (2 \cdot 0 + 1)\, D_2 g(1, -1) + 1$$

$$\Rightarrow D_2 g(1, -1) = -1$$

Additional Problem 1: Show that the Implicit Function Theorem implies (the following version of) the Inverse Function Theorem:

Inverse Function Theorem: Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. If $\det(f'(a)) \neq 0$ for some a, there is an open neighborhood U of a and an open neighborhood V of $f(a)$ and $g : V \to U$ such that $f(g(y)) = y$ for all y. Moreover

$$g'(f(a)) = (f'(a))^{-1}$$

Note: Hence, combined with the proof given in lecture, the Inverse and Implicit function theorems are equivalent.

Implicit Function Theorem:

Suppose $F : \mathbb{R}^{n+m} \to \mathbb{R}^m$ is C^1 and $F(x_0, y_0) = 0$ for some (x_0, y_0).

If $\det F_y(x_0, y_0) \neq 0$ then there is an open neighborhood U of (x_0, y_0) and an open neighborhood W of x_0 and a function $G : W \to \mathbb{R}^m$ differentiable at x_0 such that

$$\{(x, y) \in U \mid F(x, y) = 0\} = \{(x, G(x)) \mid x \in W\}$$

Moreover $G'(x_0) = -(F_y(x_0, y_0))^{-1} F_x(x_0, y_0)$

In other words, if the derivative with respect to the variable you want to solve for is invertible, then the equation $F(x, y) = 0$ is locally the graph of a function $y = G(x)$.

Additional Problem 1: Apply the Implicit Function theorem to $F : \mathbb{R}^{n+n} \to \mathbb{R}^n$ given by $F(x, y) = y - f(x)$ and $(x_0, y_0) = (a, f(a))$ and let $g(y) = G(y)$, where G is from the Implicit Function Theorem. Careful that here the roles of x and y are switched

Considering $F : \mathbb{R}^{n+n} \to \mathbb{R}^n$ $\quad F(x, y) = y - f(x) \in \mathbb{R}^n$

$(x_0, y_0) = (a, f(a))$

$F(x_0, y_0) = F(a, f(a)) = f(a) - f(a) = 0$

$F_x(x_0, y_0) = \lim_{t \to 0} \frac{F(x_0 + t, y_0) - F(x_0, y_0)}{t}$

$= \lim_{t \to 0} \frac{f(a) - f(a+t)}{t} = f'(a)$

Because $\det(f'(a)) \neq 0$ $\quad \det F_x(x_0, y_0) \neq 0$

Then there is an open neighborhood W of $(a, f(a))$, and an open neighborhood V of $f(a)$ and a function $g : V \to \mathbb{R}^n$ such that

$\{(x, y) \in W \mid F(x, y) = 0\} = \{(g(y), y) \mid y \in V\}$

which means there is an open neighborhood V of $f(a)$ s.t.

$F(g(y), y) = y - f(g(y)) = 0$, $\iff y = f(g(y))$ when $y \in V$

And $(g(y), y) \in W$. W open, take U as the first n dim of W, then $g(y) \in U$, U open, $g : V \to U$

Also $g'(y_0) = -(F_x(x_0, y_0))^{-1} F_y(x_0, y_0)$

$= -(f'(a))^{-1} \cdot 1$

$= -(f'(a))^{-1}$

$F_y(x_0, y_0) = \lim_{t \to 0} \frac{F(x_0, y_0 + t) - F(x_0, y_0)}{t}$

$= \lim_{t \to 0} \frac{y_0 + t - y_0}{t}$

$= 1$

Then $g'(f(a)) = -(f'(a))^{-1}$