# MATH 409 – HOMEWORK 8

**Reading:** Sections 17 and 18

- **Section 17:** 8 (see Note), 12, 13, 14, AP1, AP2, AP3
- **Section 18:** 9, 10, 12, AP4, (Optional: AP5)

**Notes:** For Problem 8(b), please do this directly, without using (a) (use the definition of min)

**Additional Problem 1:** Use the $\epsilon - \delta$ definition of continuity to prove

(a) $f(x) = |x|$ is continuous

(b) $f(x) = \frac{1}{x}$ is continuous at $x_0$, for all $x_0 \neq 0$

(c) $f(x) = \sqrt{x}$ is continuous at $x_0$, for all $x_0 > 0$

**Definition:**

If $f : \mathbb{R} \to \mathbb{R}$, and $U$ is any subset of $\mathbb{R}$, then the **pre-image** $f^{-1}(U)$ is defined by

$$x \in f^{-1}(U) \Leftrightarrow f(x) \in U$$

This definition works for *any* function $f$, not just invertible ones!

**Example:** $f(x) = 2x + 3$, then $f^{-1}((5, 9)) = (1, 3)$ because

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$$x \in f^{-1}((5,9)) \Leftrightarrow f(x) \in (5,9) \Leftrightarrow 5 < 2x+3 < 9 \Leftrightarrow 1 < x < 3$$

**Additional Problem 2:** Calculate $f^{-1}(U)$ for the following functions $f$ and the following sets $U$

(a) $f(x) = 3x+7$, $U = (7,10)$

(b) $f(x) = x^2$, $U = (-1,4)$

(c) $f(x) = \sin(x)$, $U = (0,1)$

**Note:** Observe that in all of the examples, both $U$ and $f^{-1}(U)$ are open intervals (or unions of open intervals) This is precisely because $f$ is continuous. In fact, in topology, this is taken as the *definition* of continuity, since it only involves open sets:

**Fact: (do not prove)**

$f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if

$$U \text{ is open } \Rightarrow f^{-1}(U) \text{ is open}$$

**Additional Problem 3:** To illustrate the elegance of the above definition, let's give a quick proof of the fact that composition of continuous functions are continuous. You do not need to know the definition of open to do this problem.

(a) If $f$ and $g$ are any functions (not necessarily invertible), prove that
$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$$

(b) Use (a) and the definition above to show that if $f$ and $g$ are continuous, then $g \circ f$ is continuous

**Additional Problem 4:** Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(f(f(x))) = x$ for all $x$, then $f(x) = x$.

**Optional Additional Problem 5:** Prove that, for any function $f$ and any sets $A$ and $B$, we have

(a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(b) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

(c) $f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$

**Hints:**

**17.8(b)** Do it by cases, first assuming $f(x) \leq g(x)$ and then assuming $g(x) \leq f(x)$. Remember that I did the version with max in the following video: Max is continuous

**17.12** For (a), remember that $\mathbb{Q}$ is dense in $\mathbb{R}$ (the sequence definition), and for (b), consider $f(x) - g(x)$ and use the result in (a)

**17.13** For (a), if $x_0$ is irrational, use $\mathbb{Q}$ is dense in $\mathbb{R}$, or, if $x_0$ is rational use that $x_n = x_0 + \frac{\sqrt{2}}{n}$ is a sequence of irrational numbers converging to $x_0$. For (b), use a similar argument, except for $x_0 = 0$ where you use $|h(x)| \leq |x|$

**17.14** I'm surprised the book didn't put any hints here, because it's a nontrivial problem! First of all, if $x_0$ is rational use the sequence $x_n = x_0 + \frac{\sqrt{2}}{n}$.

To show $f$ is continuous at irrational points $x_0$, use an $\epsilon - \delta$ argument as follows:

Let $\epsilon > 0$ be given, and let $N$ be such that $\frac{1}{N} < \epsilon$. Then, choose $\delta$ so small that there are no integers in $(x_0 - \delta, x_0 + \delta)$, and now choose $\delta$ so small that there are no fractions with denominator 2 in $(x_0 - \delta, x_0 + \delta)$, and choose $\delta$ even smaller that there are no fractions with denominator 3 in $(x_0 - \delta, x_0 + \delta)$, and so on, until there are no fractions with denominator $N$ in $(x_0 - \delta, x_0 + \delta)$.

If $|x - x_0| < \delta$ and $x = \frac{p}{q}$ is rational, show $q \geq N + 1$ and conclude. And what if $x$ is irrational?

This is sometimes called the Popcorn function ☺

**18.9** The book's hint is a bit confusing in my opinion, so here's a better one: Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$. WLOG, we may assume $a_n > 0$. Since $f$ goes to $\infty$ as $x$ goes to $\infty$, there is $b > 0$ large enough such that $f(b) > 0$, and since $f$ goes to $-\infty$ as $x$ goes to $-\infty$, there is $a < 0$ such that $f(a) < 0$. There is no need to prove those statements since we haven't defined limits yet.

**18.10** Argue in cases, if $f(1) > f(0)$ or $f(1) = f(0)$ or $f(1) < f(0)$

**18.12** Suppose $a < b$, then if $0 < a < b$, then since $f$ is continuous on $[a, b]$ we can just apply the IVT, and similarly if $a < b < 0$, so the interesting case is $a \leq 0 \leq b$. WLOG, assume $b > 0$. Apply the IVT first on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ to find $x$ such that $\sin(x) = c$ and then let $x_0 = \frac{1}{x + 2\pi n}$ for $n$ so large that $x_0 \in (0, b)$

**AP 1** For $(b)$, this is similar to the part in the lecture where I showed that $\frac{1}{f}$ is continuous: you have to assume $|x - x_0| < \frac{|x_0|}{2}$ and solve for

$|x|$ using the reverse triangle inequality. For (c), multiply $\sqrt{x} - \sqrt{x_0}$ by $\frac{\sqrt{x}+\sqrt{x_0}}{\sqrt{x}+\sqrt{x_0}}$.

**AP 3(a)** If $x \in (g \circ f)^{-1}(U)$, then $(g \circ f)(x) \in U$, so $g(f(x)) \in U$, but then what can you tell me about $f(x)$? and then what can you tell me about $x$? Likewise, if $x \in f^{-1}\left(g^{-1}(U)\right)$, what can you tell me about $f(x)$? What can you tell me about $g(f(x))$ ? So what can you tell me about $x$?

**AP 4** First show that $f$ must be one-to-one. For this suppose $f(x) = f(y)$ and apply $f$ twice to this equation. Therefore, $f$ must be increasing or decreasing (see lecture or book). But if $f$ is decreasing, suppose $x < y$ and apply $f$ three times to get a contradiction. Hence $f$ is increasing. Now if $f(x) \neq x$ for some $x$, then either $f(x) > x$ or $f(x) < x$. In both cases, apply $f$ twice to get a contradiction.

**Note:** You can find solutions to this problem in the following video: Press $fff$ to pay respects