HOMEWORK 8 – AP SOLUTIONS

AP 1

(a) Let $\epsilon > 0$ be given, let $\delta = \epsilon$, then if $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| = ||x| - |x_0|| \leq |x - x_0| < \epsilon \checkmark$$

Hence $f(x) = |x|$ is continuous

(b) **STEP 1:** Scratchwork

$$|f(x) - f(x_0)| = \left|\frac{1}{x} - \frac{1}{x_0}\right| = \left|\frac{x_0 - x}{x x_0}\right| = \frac{|x - x_0|}{|x_0|\,|x|}$$

Now if $|x - x_0| < \frac{|x_0|}{2}$, then

$$|x - x_0| \geq ||x| - |x_0|| \geq -(|x| - |x_0|) = |x_0| - |x|$$

And therefore

$$|x_0| - |x| < \frac{|x_0|}{2} \Rightarrow |x| > |x_0| - \frac{|x_0|}{2} = \frac{|x_0|}{2}$$

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Hence $\frac{1}{|x|} < \frac{2}{|x_0|}$, and therefore:

$$\begin{aligned} |f(x) - f(x_0)| &= \left(\frac{|x - x_0|}{|x_0|}\right)\left(\frac{1}{|x|}\right) \\ &\leq \left(\frac{|x - x_0|}{|x_0|}\right)\left(\frac{2}{|x_0|}\right) \\ &= |x - x_0| \left(\frac{2}{|x_0|^2}\right) \\ &< \epsilon \end{aligned}$$

Which gives $|x - x_0| = \frac{\epsilon |x_0|^2}{2}$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, let $\delta = \min\left\{\frac{|x_0|}{2}, \frac{\epsilon |x_0|^2}{2}\right\}$, then if $|x - x_0| < \delta$, then

$$\begin{aligned} |f(x) - f(x_0)| &= \left(\frac{|x - x_0|}{|x_0|}\right)\left(\frac{1}{|x|}\right) \\ &\leq \left(\frac{|x - x_0|}{|x_0|}\right)\left(\frac{2}{|x_0|}\right) \\ &= |x - x_0| \left(\frac{2}{|x_0|^2}\right) \\ &< \left(\frac{\epsilon |x_0|^2}{2}\right)\left(\frac{2}{|x_0|^2}\right) \\ &= \epsilon \checkmark \end{aligned}$$

Hence f is continuous at x_0

(c) **STEP 1:** Scratchwork

$$\begin{aligned}
|f(x) - f(x_0)| &= \left|\sqrt{x} - \sqrt{x_0}\right| \\
&= \left|\left(\sqrt{x} - \sqrt{x_0}\right)\left(\frac{\sqrt{x} + \sqrt{x_0}}{\sqrt{x} + \sqrt{x_0}}\right)\right| \\
&= \left|\left(\sqrt{x}\right)^2 - \left(\sqrt{x_0}\right)^2\right| \left|\frac{1}{\sqrt{x} + \sqrt{x_0}}\right| \\
&= |x - x_0| \left(\frac{1}{\sqrt{x} + \sqrt{x_0}}\right) \\
&\leq |x - x_0| \left(\frac{1}{\sqrt{x_0}}\right) \\
&< \epsilon
\end{aligned}$$

Which gives $|x - x_0| < \left(\sqrt{x_0}\right)\epsilon$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, let $\delta = \left(\sqrt{x_0}\right)\epsilon$, then if $|x - x_0| < \delta$, then

$$\begin{aligned}
|f(x) - f(x_0)| &= |x - x_0| \left(\frac{1}{\sqrt{x} + \sqrt{x_0}}\right) \\
&\leq |x - x_0| \left(\frac{1}{\sqrt{x_0}}\right) \\
&< \frac{\left(\sqrt{x_0}\right)\epsilon}{\sqrt{x_0}} \\
&= \epsilon
\end{aligned}$$

Hence f is continuous at x_0

AP 2

(a)

$$\begin{aligned} x \in f^{-1}((7,10)) &\Leftrightarrow f(x) \in (7,10) \\ &\Leftrightarrow 7 < 3x + 7 < 10 \\ &\Leftrightarrow 0 < 3x < 3 \\ &\Leftrightarrow 0 < x < 1 \end{aligned}$$

Hence $f^{-1}(U) = (0,1)$

(b)

$$\begin{aligned} x \in f^{-1}((-1,4)) &\Leftrightarrow f(x) \in (-1,4) \\ &\Leftrightarrow -1 < x^2 < 4 \\ &\Leftrightarrow -2 < x < 2 \end{aligned}$$

Hence $f^{-1}(U) = (-2,2)$

(c)

$$\begin{aligned} x \in f^{-1}((0,1)) &\Leftrightarrow f(x) \in (0,1) \\ &\Leftrightarrow 0 < \sin(x) < 1 \\ &\Leftrightarrow x \in \left(2\pi m, 2\pi m + \frac{\pi}{2}\right) \cup \left(2\pi m + \frac{\pi}{2}, (2m+1)\pi\right), m \in \mathbb{Z} \end{aligned}$$

Hence

$$f^{-1}((0,1)) = \bigcup_{m \in \mathbb{Z}} \left(2\pi m, 2\pi m + \frac{\pi}{2}\right) \cup \left(2\pi m + \frac{\pi}{2}, (2m+1)\pi\right)$$

AP 3

(a)

$$\begin{aligned} x \in (g \circ f)^{-1}(U) &\Leftrightarrow (g \circ f)(x) \in U \\ &\Leftrightarrow g(f(x)) \in U \\ &\Leftrightarrow f(x) \in g^{-1}(U) \\ &\Leftrightarrow x \in f^{-1}\left(g^{-1}(U)\right) \end{aligned}$$

(b) Suppose U is open, then since g is continuous, $g^{-1}(U)$ is open, and hence, since f is continuous, $f^{-1}\left(g^{-1}(U)\right)$ is open, and therefore

$$(g \circ f)^{-1}(U) = f^{-1}\left(g^{-1}(U)\right) \text{ is open } \checkmark$$

Hence $g \circ f$ is continuous $\square$

AP 4

First, let's show f is one-to-one. Suppose $f(x) = f(y)$, then $f(f(x)) = f(f(y))$, so $f(f(f(x))) = f(f(f(y)))$, hence $x = y$ ✓

Now since f is continuous and one-to-one on $\mathbb{R}$, f must be either increasing or decreasing.

But if f were decreasing, then if $a < b$, then $f(a) > f(b)$ so $f(f(a)) < f(f(b))$ and hence $f(f(f(a))) > f(f(f(b)))$, hence $a > b \Rightarrow\Leftarrow$

Hence f is increasing.

Now suppose $f(x) \neq x$, then either $f(x) < x$ or $f(x) > x$.

But if $f(x) < x$, then $f(f(x)) < f(x)$ so $f(f(f(x))) < f(f(x))$ and therefore $x < f(f(x))$, and so we get:

$$x < f(f(x)) < f(x) < x \Rightarrow\Leftarrow$$

And we get a similar contradiction if $f(x) > x \Rightarrow\Leftarrow$.

Therefore we must have $f(x) = x$ for all x $\square$

AP 5

(a)

$$\begin{aligned} x \in f^{-1}(A \cup B) &\Leftrightarrow f(x) \in A \cup B \\ &\Leftrightarrow (f(x) \in A) \text{ or } (f(x) \in B) \\ &\Leftrightarrow \left(x \in f^{-1}(A)\right) \text{ or } \left(x \in f^{-1}(B)\right) \\ &\Leftrightarrow x \in f^{-1}(A) \cup f^{-1}(B) \end{aligned}$$

(b)

$$\begin{aligned} x \in f^{-1}(A \cap B) &\Leftrightarrow f(x) \in A \cap B \\ &\Leftrightarrow (f(x) \in A) \text{ and } (f(x) \in B) \\ &\Leftrightarrow \left(x \in f^{-1}(A)\right) \text{ and } \left(x \in f^{-1}(B)\right) \\ &\Leftrightarrow x \in f^{-1}(A) \cap f^{-1}(B) \end{aligned}$$

(c)

$$\begin{aligned} x \in f^{-1}(A^c) &\Leftrightarrow f(x) \in A^c \\ &\Leftrightarrow f(x) \notin A \\ &\Leftrightarrow \text{ Not } \ (f(x) \in A) \\ &\Leftrightarrow \text{ Not } \ \left(x \in f^{-1}(A)\right) \\ &\Leftrightarrow x \notin f^{-1}(A) \\ &\Leftrightarrow x \in \left(f^{-1}(A)\right)^c \end{aligned}$$