HOMEWORK 8 – SELECTED BOOK SOLUTIONS

17.8(b)

Case 1: $f(x) \leq g(x)$

In this case $\min(f, g) = f$, but, since $-f \geq -g$, we have $\max(-f, -g) = -f$, and therefore

$$-\max(-f, -g) = -(-f) = f = \min(f, g)\checkmark$$

Case 1: $g(x) \leq f(x)$

In this case $\min(f, g) = g$, but, since $-g \geq -f$, we have $\max(-f, -g) = -g$, and therefore

$$-\max(-f, -g) = -(-g) = f = \min(f, g)\checkmark \quad \square$$

17.12(a)

Let $x \in (a, b)$, then since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is a sequence (x_n) of rational numbers converging to x. But then, on the other hand, $f(x_n) \to f(x)$ since f is continuous. And, on the other hand, since $f(x_n) = 0$ (since x_n is rational), we get $f(x) = 0$ $\square$

17.12(b)

Consider $h(x) = f(x) - g(x)$. Then h is continuous, being the difference of continuous functions. Moreover, if r is rational, then $h(r) =$

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$f(r) - g(r) = 0 - 0 = 0$, therefore by part (a), we have $h(x) = 0$ for all x, and so $f(x) - g(x) = 0$, that is $f(x) = g(x)$ for all x □

17.13(a)

Case 1: If x is irrational, then since $\mathbb{Q}$ is dense in $\mathbb{R}$, let x_n be a sequence of rational numbers converging to x, but then

$$f(x_n) = 1 \nrightarrow 0 = f(x)$$

Therefore f is discontinuous at irrational x

Case 2: If x is rational, let $x_n = x + \frac{\sqrt{2}}{n}$. Then x_n is irrational (otherwise $\sqrt{2} = n(x_n - x)$ would be rational), and $x_n \to x$, but

$$f(x_n) = 0 \nrightarrow 1 = f(x)$$

Hence f is discontinuous at rational x

17.13(b)

First of all, let's show that h is continuous at $x = 0$. Let $\epsilon > 0$ be given and let $\delta = \epsilon > 0$, then if $|x| < \delta$, then

$$|h(x) - h(0)| = |h(x)| \leq |x| < \epsilon \checkmark$$

(Here we used $|h(x)| \leq |x|$, which you can show by cases)

Therefore h is continuous at x

Now if $x \neq 0$ is irrational, then, since $\mathbb{Q}$ is dense in $\mathbb{R}$, let x_n be a sequence of rational numbers converging to x, but then

$$h(x_n) = x_n \to x \neq 0 = h(x)$$

Hence h is discontinuous at irrational x

And if $x \neq 0$ is rational, then let $x_n = x + \frac{\sqrt{2}}{n}$, which is a sequence of irrational numbers converging to x, but then

$$h(x_n) = 0 \to 0 \neq x = h(x)$$

Hence h is discontinuous at rational x

17.14

STEP 1: Suppose x_0 is rational, and let's show that f is discontinuous at x_0. let $x_n = x + \frac{\sqrt{2}}{n}$. Then x_n is a sequence of irrational numbers converging to x_0, but then

$$f(x_n) = 0 \nrightarrow f(x_0) \neq 0$$

And therefore f is discontinuous at x_0.

And if $x_0 = 0$, then let $x_n = \frac{1}{n} \to 0$, then $x_n \to x_0$ but then

$$f(x_n) = \frac{1}{n} \to 0 \neq 1 = f(0)$$

Therefore f is also discontinuous at 0

Hence f is discontinuous at all the rational numbers

STEP 2: Suppose x_0 is irrational, and let $\epsilon > 0$ be given.

Then let N be such that $\frac{1}{N} < \epsilon$

Now let δ be so small such that there are no integers in $(x_0 - \delta, x_0 + \delta)$ and no fractions with denominator 2, no fractions with denominator

$3, \dots$ and no fractions with denominator $\frac{1}{N}$.

Then, if $|x - x_0| < \delta$ and x is irrational, then

$$|f(x) - f(x_0)| = |0 - 0| = 0 < \epsilon \checkmark$$

And if $x = \frac{p}{q}$ is rational (where p and q have no common factors and $q > 0$), then by the above we must have $q > N$, and so

$$|f(x) - f(x_0)| = \left| \frac{1}{q} - 0 \right| = \frac{1}{q} < \frac{1}{N} < \epsilon$$

Therefore f is continuous at x_0 ✓

And hence f is continuous at all irrational numbers and discontinuous at all rational numbers. □

18.9

Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ where $n \geq 1$ is odd and $a_n \neq 0$.

Assume WLOG that $a_n > 0$ (consider $-f$ otherwise)

Notice

$$\begin{aligned} \lim_{x \to \infty} f(x) &= \lim_{x \to \infty} x^n \left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \cdots + a_n \right) \\ &= \infty \times a_0 \\ &= \infty \end{aligned}$$

And therefore in particular there is $b > 0$ large enough such that $f(b) > 0$

On the other hand,

$$\begin{aligned}\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} x^n\left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \cdots + a_n\right)\\ &= (-\infty)^n \times a_0\\ &= -\infty \times a_0 \text{ (Since } n \text{ is odd)}\\ &= -\infty\end{aligned}$$

And therefore there is $a < 0$ large enough such that $f(a) < 0$

Since f is continuous on $[a, b]$ (see for example 17.5(b)), by the Intermediate Value Theorem with $c = 0$, there is $x_0 \in [a, b]$ such that $f(x_0) = 0$, so f has a root in (a, b) □

18.10

If $f(0) = f(1)$, then we're done (just let $x = 0$ or $x = 1$), so WLOG assume $f(0) < f(1)$, since the case $f(0) > f(1)$ is similar.

Let $g(x) = f(x + 1) - f(x)$, which is continuous on $[0, 1]$ since f is continuous

Then $g(0) = f(1) - f(0) > 0$ and $g(1) = f(2) - f(1) = f(0) - f(1) < 0$

Therefore by the Intermediate Value Theorem with $c = 0$, there is x_0 in $[0, 1]$ such that $g(x_0) = 0$, that is $f(x_0 + 1) - f(x_0) = 0$ so $f(x_0 + 1) = f(x_0)$.

Therefore if you let $x = x_0 \in [0, 1]$ and $y = x_0 + 1 \in [1, 2]$, then by the above we get $|x - y| = |x_0 - (x_0 + 1)| = |-1| = 1$ and $f(x) = f(x_0) = f(x_0 + 1) = f(y)$ ✓ □

18.12

The fact that f is discontinuous at 0 was shown in Exercise 17.10(b)

Let's show that f has the intermediate value property:

Suppose $a < b$ and c is a value between $f(a)$ and $f(b)$

Now if $a < b < 0$ or $0 < a < b$, then we're done because f would be continuous on $[a, b]$ and hence have the intermediate value property.

Therefore assume $a \leq 0 \leq b$ and since $a \neq b$, we must have either $a < 0$ or $b > 0$. WLOG, assume $b > 0$

Since $\sin(x)$ is continuous on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ and c is (in particular) between -1 and 1, there is x between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ such that $\sin(x) = c$.

Now let $x_0 =: \frac{1}{x+2\pi n}$, where n is so large such that $x_0 \in (0, b)$ (we can do that since x_0 goes to 0 as n goes to ∞). Then:

$$f(x_0) = \sin\left(\frac{1}{x_0}\right) = \sin(x + 2\pi n) = \sin(x) = c\checkmark \quad \square$$