27. Put $f(0,0)=0$, and

$$f(x,y)=\frac{xy(x^2-y^2)}{x^2+y^2}$$

if $(x,y)\neq(0,0)$. Prove that

(*a*) $f, D_1 f, D_2 f$ are continuous in R^2;

(*b*) $D_{12}f$ and $D_{21}f$ exist at every point of R^2, and are continuous except at $(0,0)$;

(*c*) $(D_{12}f)(0,0)=1$, and $(D_{21}f)(0,0)=-1$.

(c) $D_1 f(x,y)=\dfrac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}$

$D_2 f(x,y)=\dfrac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2}$

Then $D_1 f(0,y)=-y$, $D_2 f(x,0)=x$

By (a) $D_1 f$ and $D_2 f$ continuous in $\mathbb{R}^2$

$D_1 f(0,0)=\lim_{y\to0} D_1 f(0,y)=0$

$D_2 f(0,0)=\lim_{x\to0} D_2 f(x,0)=0$

$D_{12} f(0,0)=\lim_{y\to0}\dfrac{D_1 f(0,y)-D_1 f(0,0)}{y}=\lim_{y\to0}\dfrac{-y}{y}=-1$

$D_{21} f(0,0)=\lim_{x\to0}\dfrac{D_2 f(x,0)-D_2 f(0,0)}{x}=\lim_{x\to0}\dfrac{x}{x}=1$

which ends the proof.

28. For $t \geq 0$, put

$$\varphi(x,t) = \begin{cases} x & (0 \leq x \leq \sqrt{t}) \\ -x + 2\sqrt{t} & (\sqrt{t} \leq x \leq 2\sqrt{t}) \\ 0 & \text{(otherwise)}, \end{cases}$$

and put $\varphi(x,t) = -\varphi(x,|t|)$ if $t < 0$.

Show that φ is continuous on R^2, and

$$(D_2\varphi)(x,0) = 0$$

for all x. Define

$$f(t) = \int_{-1}^{1} \varphi(x,t)\,dx.$$

Show that $f(t) = t$ if $|t| < \frac{1}{4}$. Hence

$$f'(0) \neq \int_{-1}^{1} (D_2\varphi)(x,0)\,dx.$$

1) φ is continuous:

we only need to verify it continuous on boundary $x=0$; $x=\sqrt{t}$; $x=2\sqrt{t}$.

$$\lim_{x\to 0^+} \varphi(x,t) = \lim_{x\to 0^+} x = 0 = \lim_{x\to 0^-} x$$

Because $\varphi(x,t) \equiv 0$ when $x<0$

$$\lim_{x\to\sqrt{t}^-} \varphi(x,t) = \lim_{x\to\sqrt{t}^-} x = \sqrt{t} = -\sqrt{t}+2\sqrt{t} = \lim_{x\to\sqrt{t}^+} \varphi(x,t)$$

$$\lim_{x\to 2\sqrt{t}^-} \varphi(x,t) = \lim_{x\to 2\sqrt{t}^-} (-x+2\sqrt{t}) = -2\sqrt{t}+2\sqrt{t} = 0 = \lim_{x\to 2\sqrt{t}^+} \varphi(x,t)$$

note $\varphi(x,t) \equiv 0$ when $x > 2\sqrt{t}$

2) $(D_2\varphi)(x,0)$

$$= \lim_{t\to 0} \frac{\varphi(x,t) - \varphi(x,0)}{t}$$

when $x<0$, $(D_2\varphi)(x,0) = 0$

when $x>0$. when $0<|t|<\frac{x^2}{4}$, $x > 2\sqrt{|t|}$ then $\varphi(x,t) = 0$

then $$(D_2\varphi)(x,0) = \lim_{t\to 0} \frac{\varphi(x,t)-\varphi(x,0)}{t} = \lim_{\substack{|t|<\frac{x^2}{4} \\ t\to 0}} \frac{\varphi(x,t)-\varphi(x,0)}{t} = 0$$

thus $(D_2\varphi)(x,0) = 0$

3) $|t| < \frac{1}{4}$ then $t < \frac{x^2}{4}$

if $t \geq 0$ $f(t) = \int_{-1}^{1} \varphi(x,t)\,dx$

$$= \int_0^{\sqrt{t}} x\,dx + \int_{\sqrt{t}}^{2\sqrt{t}} (-x+2\sqrt{t})\,dx$$

$$= \frac{t}{2} + -\frac{x^2}{2}\Big|_{\sqrt{t}}^{2\sqrt{t}} + \sqrt{t}\cdot 2\sqrt{t}$$

$$= \frac{t}{2} + -2t + \frac{t}{2} + 2t$$

$$= t$$

if $t<0$ $f(t) = \int_{-1}^{1} -\varphi(x,-t)\,dx$

$$= -\int_{-1}^{1} \varphi(x,-t)\,dx = -(-t) = t$$

Then $f'(0) = 1 \neq 0 = \int_{-1}^{1} (D_2\varphi)(x,0)\,dx$

Additional Problem 1: Let C be the middle-thirds Cantor set, as defined in section 2.44 in Chapter 2 of Rudin. Show that $m_\star(C) = 0$ (and hence C is measurable).

2.44 The Cantor set The set which we are now going to construct shows that there exist perfect sets in R^1 which contain no segment.

Let E_0 be the interval $[0, 1]$. Remove the segment $(\frac{1}{3}, \frac{2}{3})$, and let E_1 be the union of the intervals

$$[0, \tfrac{1}{3}]\ [\tfrac{2}{3}, 1].$$

Remove the middle thirds of these intervals, and let E_2 be the union of the intervals

$$[0, \tfrac{1}{9}], [\tfrac{2}{9}, \tfrac{3}{9}], [\tfrac{6}{9}, \tfrac{7}{9}], [\tfrac{8}{9}, 1].$$

Continuing in this way, we obtain a sequence of compact sets E_n, such that

(*a*) $E_1 \supset E_2 \supset E_3 \supset \cdots$;
(*b*) E_n is the union of 2^n intervals, each of length 3^{-n}.

The set

$$P = \bigcap_{n=1}^{\infty} E_n$$

is called the *Cantor set*. P is clearly compact, and Theorem 2.36 shows that P is not empty.

For every i

E_i as defined is the union of 2^n closed intervals, each of length 3^{-n}.

By the property of exterior measure, and these 2^n intervals are disjoint,

thus $m_*(E_i) = 2^n \cdot 3^{-n} = \left(\frac{2}{3}\right)^n$

Because $E_1 \supset E_2 \supset \cdots \supset E_n \cdots \supset P$

By **Property 1:** (Monotonicity) If $E_1 \subseteq E_2$ then $m_\star(E_1) \le m_\star(E_2)$

Then $m_*(E_1) \geq m_*(E_2) \geq \cdots \geq m_*(E_n) \geq \cdots \geq m_*(P)$

take $\varepsilon > 0$, Because $\lim_{n \to \infty} \left(\frac{2}{3}\right)^n = 0$,

exists $n \in \mathbb{N}$ s.t. $m_*(E_n) = \left(\frac{2}{3}\right)^n < \varepsilon$

Then $m_*(P) < \varepsilon$

Thus $m_*(P) = 0$

Additional Problem 2: Suppose $E_1, E_2, \dots$ is a countable collection of measurable subsets of $\mathbb{R}^d$ (see hints for definitions)

(a) Show that if E_k increases to E then $m(E) = \lim_{N\to\infty} m(E_N)$

(b) Show that if E_k decreases to E and $m(E_k) < \infty$ for some k then $m(E) = \lim_{N\to\infty} m(E_N)$

(c) Show that (b) is false if we don't assume that $m(E_k) < \infty$

Definition: E_k **increases** to E and $E_k \nearrow E$ if $E_k \subseteq E_{k+1}$ and $E = \bigcup_{k=1}^{\infty} E_k$ and we say that E_k **decreases** to E and $E_k \searrow E$ if $E_k \supseteq E_{k+1}$ and $E = \bigcap_{k=1}^{\infty} E_k$

Property 5: If $E_1, E_2, \dots$ are disjoint **measurable sets** and $E = \bigcup_{j=1}^{\infty} E_j$ then

$$m(E) = \sum_{j=1}^{\infty} m(E_j)$$

(a) let

$G_1 = E_1$, $G_2 = E_2 - E_1$, $G_k = E_k - E_{k-1}$,

Because $E_k \subseteq E_{k+1}$, we know that G_i are disjoint measurable sets

So $E_N = \bigcup_{i=1}^{N} E_i = \bigcup_{i=1}^{N} G_i$, $m(E_N) = \sum_{i=1}^{N} m(G_i)$

Then $E = \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{k} G_i = \bigcup_{i=1}^{\infty} G_i$

Then By Property 5

$m(E) = \sum_{i=1}^{\infty} m(G_i) = \lim_{N\to\infty} \sum_{i=1}^{N} m(G_i) = \lim_{n\to\infty} m(E_N)$ #

(b) WLOG, we assume $m(E_1) < \infty$

let $G_k = E_k - E_{k+1}$

$E_1 = E \cup \bigcup_{k=1}^{\infty} G_k$

By Property 5, $m(E_1) = m(E) + \sum_{k=1}^{\infty} m(G_k)$ (I)

we have $\sum_{k=1}^{N} G_k = E_1 - E_N$, then

$m\left(\sum_{k=1}^{N} G_k\right) = \sum_{k=1}^{N} m(G_k) = m(E_1) - m(E_N)$ (II)

From (I) & (II) we have

$m(E_1) = m(E) + \lim_{N\to\infty} \sum_{k=1}^{N} m(G_k)$ By (II)

$= m(E) + \lim_{N\to\infty} (m(E_1) - m(E_N))$

$m(E_1) - m(E_1) + \lim_{N\to\infty} m(E_N) = m(E) \implies m(E) = \lim_{N\to\infty} m(E_N)$ #

(c) Considering $E_n = (n, \infty)$, $m(E_n) = \infty$

But $E = \bigcap_{n=1}^{\infty} E_n = \emptyset$ and $m(E) = 0 \neq \infty = \lim_{n\to\infty} m(E_n)$

#

Additional Problem 3: [The Borel-Cantelli Lemma] Suppose $\{E_k\}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty$$

Define $E = \{x \in \mathbb{R}^d \mid x \in E_k \text{ for infinitely many } k\}$

(a) Show that E is measurable

(b) Show $m(E) = 0$

(a) CLAIM: $E = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$

1) $E \subset \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$

any $x \in E$, because $x \in E_k$ for infinitely many k

For any $N \in \mathbb{N}$ $x \in \bigcup_{k=N}^{\infty} E_k$

(otherwise x at most exists in $E_1, \cdots E_{N-1}$, finitely many sets)

Then, $x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$

2) $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \subset E$

any $x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k = \left(\bigcup_{k=1}^{\infty} E_k\right) \cap \left(\bigcup_{k=2}^{\infty} E_k\right) \cap \cdots \cap \left(\bigcup_{k=n}^{\infty} E_k\right) \cap \cdots$

By assuming contrary, we assume

x only exists in finite many E_k,

Then $\exists N \in \mathbb{N}$ s.t. $x \notin E_n$ for all $n \geq N$

then $x \notin \bigcup_{k=N}^{\infty} E_k$

then $x \notin \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$ which leads to contradiction.

So x exists in infinite many E_k, thus $x \in E$.

Then E is measurable, because Lebesgue measurable subsets of $\mathbb{R}^d$ forms a σ-Algebra.

(b) $m\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m(E_k) < \infty$

notice that

$\left\{\bigcup_{k=n}^{\infty} E_k\right\}$ is non-increasing

$$\bigcup_{k=1}^{\infty} E_k \supseteq \bigcup_{k=2}^{\infty} E_k \supseteq \bigcup_{k=3}^{\infty} E_k \cdots \supseteq \bigcup_{k=N}^{\infty} E_k \cdots \supseteq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

then $m\left(\cap \cup E\right) = \lim_{n\to\infty} m\left(\bigcup_{k=n}^{\infty} E_k\right)$

$$\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) \quad (*)$$

We claim that $\lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) = 0$

proof: we know that $\sum_{k=1}^{\infty} m(E_k) < \infty$, i.e. $\lim_{n\to\infty} \sum_{k=1}^{n} m(E_k)$ exists

then take any $\varepsilon > 0$

$\exists N \in \mathbb{N}$ s.t. $\left|\sum_{k=1}^{\infty} m(E_k) - \sum_{k=1}^{N-1} m(E_k)\right| < \varepsilon$

thus $\left|\sum_{k=N}^{\infty} m(E_k)\right| < \varepsilon$

which also means $\lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) = 0$

Then by $(*)$ and our claim,

$0 \leq m(E) \leq 0 \Rightarrow m(E) = 0$