MATH S4062 – HOMEWORK 8

- **Chapter 9:** 27(c), 28

Please also do the Additional Problems below:

Additional Problem 1: Let C be the middle-thirds Cantor set, as defined in section 2.44 in Chapter 2 of Rudin. Show that $m_{\star}(C) = 0$ (and hence C is measurable).

Additional Problem 2: Suppose $E_1, E_2, \dots$ is a countable collection of measurable subsets of $\mathbb{R}^d$ (see hints for definitions)

(a) Show that if E_k increases to E then $m(E) = \lim_{N\to\infty} m(E_N)$

(b) Show that if E_k decreases to E and $m(E_k) < \infty$ for some k then $m(E) = \lim_{N\to\infty} m(E_N)$

(c) Show that (b) is false if we don't assume that $m(E_k) < \infty$

Additional Problem 3: [The Borel-Cantelli Lemma] Suppose $\{E_k\}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty$$

$$\text{Define } E = \left\{x \in \mathbb{R}^d \mid x \in E_k \text{ for infinitely many } k\right\}$$

(a) Show that E is measurable

(b) Show $m(E) = 0$

Date: Due: Sunday, August 7, 2022.

Hints:

Additional Problem 1: Show that $m_\star(C) \leq \left(\frac{2}{3}\right)^n$ for all n

Additional Problem 2:

Definition: E_k **increases** to E and $E_k \nearrow E$ if $E_k \subseteq E_{k+1}$ and $E = \bigcup_{k=1}^{\infty} E_k$ and we say that E_k **decreases** to E and $E_k \searrow E$ if $E_k \supseteq E_{k+1}$ and $E = \bigcap_{k=1}^{\infty} E_k$

For (a), notice that if $G_1 = E_1, G_2 = E_2 - E_1$ and in general $G_k = E_k - E_{k-1}$ then $E = \bigcup_{k=1}^{\infty} G_k$ (where the union is disjoint)

For (b), assume WLOG that $m(E_1) < \infty$, let $G_k = E_k - E_{k+1}$ so that $E_1 = E \cup \bigcup_{k=1}^{\infty} G_k$ and calculate $m(E_1)$

For (c), consider $E_n = (n, \infty)$

Additional Problem 3: Show that $E = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$ and use the previous problem.

Aside: E is sometimes called $\limsup_{k\to\infty} E_k$ by similarity with limsup of sequences