MATH 409 – HOMEWORK 9

Reading: Sections 19 and 20. In section 19, ignore Example 3 and Discussion 19.3 and the proof of Theorem 19.5. I will go through section 20 fairly quickly, because it's very similar to the sections on continuity. The book's notation for limits (with a^S) is a bit confusing, so it's ok if you just use my notation.

- **Section 19:** 2, 4, 5, 6, 8
- **Section 20:** 11, 20(a)(b) (see Note), AP1, AP2, AP3

Note: For Problem 20, please do this directly, without using Exercise 9.11 or Theorem 9.9. Do not use sequences.

Additional Problem 1: Use the $\epsilon - \delta$ definition of limits (or variations thereof) to show

(a) $\lim_{x\to 2} 3x + 5 = 11$

(b) $\lim_{x\to 3} x^2 = 9$

(c) $\lim_{x\to 4} \sqrt{x} = 2$

(d) $\lim_{x\to 1} \frac{1}{x} = 1$

(e) $\lim_{x\to 5^+} \frac{1}{x-5} = \infty$

(f) $\lim_{x\to\infty} 3 + \frac{2}{x^2} = 3$

Date: Due: Friday, November 12, 2021.

Additional Problem 2: Use $\epsilon - \delta$ to prove the Squeeze Theorem: If $f(x) \leq g(x) \leq h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ (finite), then $\lim_{x \to a} g(x) = L$ as well.

Additional Problem 3:

(a) Use the hints at the end to show

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

(b) Use (a) to show that

$$\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} = 0$$

(Hints on the next page)

Hints:

19.4(a) By "assume not", the book means that there is a sequence (x_n) in S such that $|f(x_n)| \to \infty$. Remember that convergent sequences are Cauchy, and that Cauchy sequences are bounded.

19.6(a) Use that $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$, hence uniformly continuous on $[0, 1]$ (which is closed and bounded)

20.11 Here you don't need to use $\epsilon - \delta$, use the usual limit laws. Of course do **not** use derivatives or L'Hôpital's rule here

20.20 For (a), it might be useful to let $\epsilon = 1$ in the definition of L_2. For (b), let $\epsilon = \frac{L_2}{2}$.

AP 1 Solutions to parts (a)-(e) can be found in the videos below. Part (f) is covered at the end of the Lecture 19 notes. The only tricky part is for (d), you can't just assume $|x - 1| < 1$, you need to assume $|x - 1| < \frac{1}{2}$

Video 1: Part (a)

Video 2: Part (b)

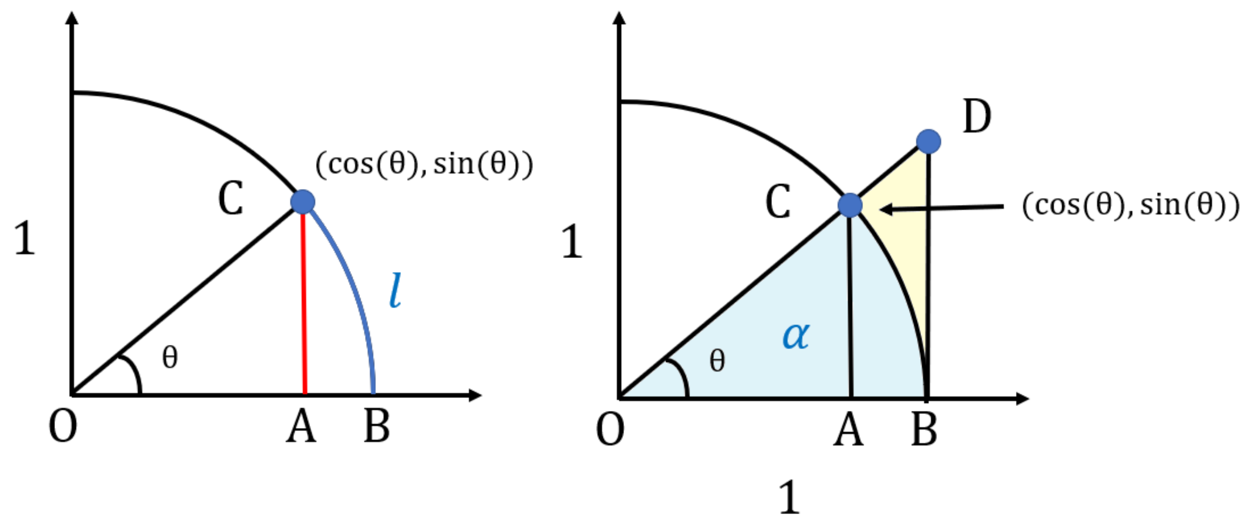
Video 3: Part (c)

Video 4: Part (d)

Video 5: Part (e)

AP 2 Solutions can be found here: Squeeze Theorem Proof

AP 3(a) First do the limit as $\theta \to 0^+$. Use the squeeze theorem. On the one hand, the height AC is less than the length l of the arc of the circle (left picture). On the other hand, the area α of the sector is less than the area of the triangle OBD (right picture). Use similar triangles. Here the quarter circle has radius 1.



Finally, for the limit as $\theta \to 0^-$, use that $\frac{\sin(\theta)}{\theta}$ is even.

The solutions can be found in the following video: $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$

AP 3(b) Multiply both sides by $\frac{\cos(\theta)+1}{\cos(\theta)+1}$