HOMEWORK 9 – AP SOLUTIONS

AP 1

(a) Let $\epsilon > 0$ be given, let $\delta = \frac{\epsilon}{3}$, then if $0 < |x-2| < \delta$, then

$$|(3x+5) - 11| = |3x-6| = 3|x-2| < 3\left(\frac{\epsilon}{3}\right) = \epsilon\checkmark$$

Hence $\lim_{x\to 2} 3x+5 = 11$

(b) **STEP 1:** Scratchwork

$$\left|x^2-4\right| = |x-2|\, x+2 < \epsilon$$

Now if $|x-2| < 1$, then

$$|x+2| = |x-2+2| \leq |x-2| + |2| < 1+2 = 3$$

And therefore

$$|x-2|\,|x+2| \leq 3|x-2| < \epsilon \Rightarrow |x-2| < \frac{\epsilon}{3}$$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, let $\delta = \min\left\{1, \frac{\epsilon}{3}\right\}$, then if $0 < |x-2| < \delta$, then $|x-2| < 1$ so $|x+2| < 3$, and so

$$\left|x^2-4\right| = |x-2|\,|x+2| \leq 3|x-2| < 3\left(\frac{\epsilon}{3}\right) = \epsilon\checkmark$$

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Hence $\lim_{x\to 2} x^2 = 4$

(c) **STEP 1:** Scratchwork

$$\begin{aligned}\left|\sqrt{x}-2\right| &= \left|\left(\sqrt{x}-2\right)\left(\frac{\sqrt{x}+2}{\sqrt{x}+2}\right)\right| \\ &= \left|\left(\sqrt{x}\right)^2 - 2^2\right|\left|\frac{1}{\sqrt{x}+2}\right| \\ &= |x-2|\left(\frac{1}{\sqrt{x}+2}\right) \\ &\leq |x-2|\left(\frac{1}{2}\right) < \epsilon\end{aligned}$$

Which gives $|x-2| < 2\epsilon$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, let $\delta = 2\epsilon$, then if $0 < |x-2| < \delta$, then

$$\left|\sqrt{x}-2\right| = |x-2|\left(\frac{1}{\sqrt{x}+2}\right) \leq |x-x_0|\left(\frac{1}{2}\right) < \frac{2\epsilon}{\frac{1}{2}} = \epsilon\checkmark$$

Hence $\lim_{x\to 4}\sqrt{x} = 2$

(d) **STEP 1:** Scratchwork

$$\left|\frac{1}{x}-1\right| = \left|\frac{1-x}{x}\right| = \frac{|x-1|}{|x|} < \epsilon$$

Now if $|x-1| < \frac{1}{2}$, then $-\frac{1}{2} < x-1 < \frac{1}{2}$ so in particular $x-1 > -\frac{1}{2}$ so $x > \frac{1}{2}$ and so $|x| > \frac{1}{2}$ and so $\frac{1}{|x|} < 2$, therefore:

$$\frac{|x-1|}{|x|} = |x-1|\left(\frac{1}{|x|}\right) \leq 2|x-1| < \epsilon \Rightarrow |x-1| < \frac{\epsilon}{2}$$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, let $\delta = \min\left\{\frac{1}{2}, \frac{\epsilon}{2}\right\}$, then if $0 < |x-1| < \delta$, then $|x-1| < \frac{1}{2}$ so $\frac{1}{|x|} < 2$, and so

$$\left|\frac{1}{x} - 1\right| = \frac{|x-1|}{|x|} \leq 2|x-1| < 2\left(\frac{\epsilon}{2}\right) = \epsilon \checkmark$$

Hence $\lim_{x\to 1} \frac{1}{x} = 1$

(e) Let $M > 0$ be given, let $\delta = \frac{1}{M}$ then if $0 < x - 5 < \delta$, then

$$\frac{1}{x-5} > \frac{1}{\delta} = \frac{1}{\frac{1}{M}} = M \checkmark$$

Hence $\lim_{x\to 5^+} \frac{1}{x-5} = \infty$

(f) **STEP 1:** Scratchwork

$$\left|3 + \frac{2}{x^2} - 3\right| = \frac{2}{x^2} < \epsilon \Rightarrow x^2 > \frac{1}{2\epsilon} \Rightarrow x > \frac{1}{\sqrt{2\epsilon}}$$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, let $N = \frac{1}{\sqrt{2\epsilon}}$, then if $x > N$, then

$$\left|3 + \frac{2}{x^2} - 3\right| = \frac{2}{x^2} < \frac{2}{\frac{2}{\epsilon}} = \epsilon \checkmark$$

Hence $\lim_{x\to\infty} 3 + \frac{2}{x^2} = 3$

AP 2

Let $\epsilon > 0$ be given.

Then since $\lim_{x\to a} f(x) = L$ there is δ_1 such that if $0 < |x-a| < \delta_1$ then $|f(x) - L| < \epsilon$, so in particular $f(x) - L > -\epsilon$

And since $\lim_{x\to a} h(x) = L$ there is δ_2 such that if $0 < |x-a| < \delta_2$ then $|h(x) - L| < \epsilon$, so in particular $h(x) - L < \epsilon$

But then, with $\delta = \min\{\delta_1, \delta_2\}$, if $0 < |x-a| < \delta$, then $|x-a| < \delta_1$ and $|x-a| < \delta_2$ and so

$$-\epsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \epsilon \Rightarrow -\epsilon < g(x) - L < \epsilon$$

Here in the middle part we used $f(x) \leq g(x) \leq h(x)$

And so $|g(x) - L| < \epsilon$ ✓

Therefore $\lim_{x\to a} g(x) = L$, as desired □

AP 3(A)

First, we'll do the limit as $\theta \to 0^+$.

Then, based on the left picture in the problem, we have $AC \leq l$

However, $AC = \sin(\theta)$ since we have a unit circle.

Moreover, the full circle with angle 2π has an arclength of $2\pi(1) = 2\pi$, so by proportionality, the arclength l with angle θ has length θ, so $l = \theta$. Therefore we get

$$AC \leq l \Rightarrow \sin(\theta) \leq \theta \Rightarrow \frac{\sin(\theta)}{\theta} \leq 1$$

On the other hand, based on the right picture in the problem, the area of the sector α is less than the area of the triangle OBD.

But by similar triangles, we have

$$\frac{AC}{BD} = \frac{OA}{OB} \Rightarrow \frac{\sin(\theta)}{BD} = \frac{\cos(\theta)}{1} \Rightarrow BC = \frac{\sin(\theta)}{\cos(\theta)}$$

And so the area of OBD is

$$\frac{1}{2} \times OB \times BD = \frac{1}{2}(1)\frac{\sin(\theta)}{\cos(\theta)} - \frac{1}{2}\left(\frac{\sin(\theta)}{\cos(\theta)}\right)$$

Now the full circle with angle 2π has an area of $\pi(1)^2 = \pi$ (half of that) so the sector α with angle α has an area of $\frac{\theta}{2}$.

Therefore we get:

$$\begin{aligned} \alpha < OBD \Rightarrow & \frac{\theta}{2} \leq \frac{1}{2}\left(\frac{\sin(\theta)}{\cos(\theta)}\right) \\ \Rightarrow & \theta \leq \frac{\sin(\theta)}{\cos(\theta)} \\ \Rightarrow & \cos(\theta) \leq \frac{\sin(\theta)}{\theta} \end{aligned}$$

Combining the two results, we get

$$\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq 1$$

And using the squeeze theorem we can conclude that $\lim_{\theta \to 0^+} \frac{\sin(\theta)}{\theta} = 1$

Now if $\theta \to 0^-$, then $-\theta \to 0^+$ so

$$\lim_{\theta \to 0^-} \frac{\sin(\theta)}{\theta} = \lim_{\theta \to 0^+} \frac{\sin(-\theta)}{-\theta} = \lim_{\theta \to 0^+} \frac{-\sin(\theta)}{-\theta} = \lim_{\theta \to 0^+} \frac{\sin(\theta)}{\theta} = 1$$

And since $\lim_{\theta \to 0^-} \frac{\sin(\theta)}{\theta} = \lim_{\theta \to 0^+} \frac{\sin(\theta)}{\theta} = 1$ we get $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$

AP 3(b)

$$\begin{aligned}
\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} &= \lim_{\theta \to 0} \frac{(\cos(\theta) - 1)(\cos(\theta) + 1)}{\theta(\cos(\theta) + 1)} \\
&= \lim_{\theta \to 0} \frac{\cos^2(\theta) - 1}{\theta(\cos(\theta) + 1)} \\
&= \lim_{\theta \to 0} \frac{-\sin^2(\theta)}{\theta(\cos(\theta) + 1)} \\
&= \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \times \frac{-\sin(\theta)}{\cos(\theta) + 1} \\
&= 1 \times \left(\frac{-\sin(0)}{\cos(0) + 1} \right) \\
&= 1 \times 0 \\
&= 0
\end{aligned}$$