HOMEWORK 9 – SELECTED BOOK SOLUTIONS

19.2(a)

STEP 1: Scratchwork

$$|f(x) - f(y)| = |3x + 11 - (3y + 11)| = |3x - 3y| = 3|x - y| < \epsilon \Rightarrow |x - y| < \frac{\epsilon}{3}$$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, let $\delta = \frac{\epsilon}{3}$, then if $|x - y| < \delta$ then

$$|f(x) - f(y)| = 3|x - y| < 3\left(\frac{\epsilon}{3}\right) = \epsilon \checkmark \quad \square$$

19.2(b)

STEP 1: Scratchwork

$$|f(x) - f(y)| = \left|x^2 - y^2\right| = |x - y|\,|x + y|$$

Now since $0 \leq x, y \leq 3$, we have $|x| \leq 3$ and $|y| \leq 3$ and so

$$|x + y| \leq |x| + |y| = 3 + 3 = 6$$

And therefore

$$|x - y|\,|x + y| \leq |x - y|\,(|x| + |y|) \leq 6|x - y| < \epsilon \Rightarrow |x - y| < \frac{\epsilon}{6}$$

STEP 2: Actual Proof

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Let $\epsilon > 0$ be given, let $\delta = \frac{\epsilon}{6}$, then if $|x - y| < \delta$ then since $0 \le x, y \le 3$ we have $|x| \le 3$ and $|y| \le 3$ and therefore

$$|f(x) - f(y)| = |x - y|\,|x + y| \le |x - y|\,(|x| + |y|) \le 6\,|x - y| < 6\left(\frac{\epsilon}{6}\right) = \epsilon\checkmark \quad \square$$

19.2(c)

STEP 1: Scratchwork

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y - x}{xy}\right| = \frac{|x - y|}{|x|\,|y|}$$

Now since $x, y \ge \frac{1}{2}$, we have $|x| \ge \frac{1}{2}$ and $|y| \ge \frac{1}{y}$ and so $\frac{1}{|x|} \le 2$ and $\frac{1}{|y|} \le 2$ and so

$$\frac{|x - y|}{|x|\,|y|} \le |x - y|\,(2)(2) = 4\,|x - y| < \epsilon \Rightarrow |x - y| < \frac{\epsilon}{4}$$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, let $\delta = \frac{\epsilon}{4}$, then if $|x - y| < \delta$ then since $x, y \ge \frac{1}{2}$ we have $|x| \le \frac{1}{2}$ and $|y| \le \frac{1}{2}$ and therefore

$$|f(x) - f(y)| = |x - y|\,|xy| \le |x - y|\,(2 \times 2) \le 4\,|x - y| < 4\left(\frac{\epsilon}{4}\right) = \epsilon\checkmark \quad \square$$

19.4

Assume f is not bounded on S, then there is a sequence (x_n) in S such that $|f(x_n)| \to \infty$ as $n \to \infty$.

But since (x_n) is in S and S is bounded, then (x_n) is bounded, and so, by the Bolzano-Weierstrass Theorem, (x_n) has a convergent subsequence (x_{n_k})

But since (x_{n_k}) is convergent, it is Cauchy, and therefore, since f is uniformly continuous, $f(x_{n_k})$ is Cauchy as well.

However, since Cauchy sequences are bounded (Lemma 10.10 in section 10), $f(x_{n_k})$ is bounded, but this contradict the fact that $|f(x_n)| \to \infty$ and therefore $|f(x_{n_k})| \to \infty$

For (b), note that even though $(0,1)$ is bounded, $f(x) = \frac{1}{x^2}$ is not bounded on $(0,1)$ (because if $|f(x)| < M$ for all x, let $x = \frac{1}{\sqrt{M}} \in (0,1)$ (if $M > 1$), then $|f(x)| = M$ which is not $< M$), therefore by (a), f is not uniformly continuous on $(0,1)$

19.5

(a) Since $\left[0, \frac{\pi}{4}\right]$ is compact and $f(x) = \tan(x)$ is continuous, f is uniformly continuous by Theorem 19.2

(b) Even though $\left[0, \frac{\pi}{2}\right)$ is bounded, $f(x) = \tan(x)$ is not bounded. And therefore f is not uniformly continuous by 19.4(a)

(c) Notice that if you let $\tilde{f}(x) = \frac{1}{x}\sin^2(x)$ if $x \neq 0$ and $\tilde{f}(0) = 0$, then $\tilde{f}(x) = \tilde{h}(x)\sin(x)$ (where $\tilde{h}(x)$ is as in Example 9) And since $\tilde{h}(x)$ is continuous on $[0,\pi]$, we get $\tilde{f}(x)$ is continuous on $[0,\pi]$. Hence $\tilde{f}$ is a continuous extension of $\frac{1}{x}\sin^2(x)$ on $[0,\pi]$, and therefore $\frac{1}{x}\sin^2(x)$ is uniformly continuous on $(0,\pi]$

(d) $\frac{1}{x-3}$ is unbounded on $(0,3)$, and therefore by 19.4(b), $\frac{1}{x-3}$ is not uniformly continuous on $(0,3)$

(e) $\frac{1}{x-3}$ is unbounded on $(3,4)$ (If $M > 1$ is given, let $x = 3 + \frac{1}{M} \in (3,4)$, then $\frac{1}{x-3} > M$, and therefore by 19.4(b), $\frac{1}{x-3}$ is not uniformly continuous on $(3,4)$ and hence not uniformly continuous on $(3,\infty)$

(f) Consider $f(x) = \frac{1}{x-3}$, then f is continuous on $(4,\infty)$ since $x - 3 \neq 0$ and moreover $f'(x) = \frac{-1}{(x-3)^2}$ and therefore

$$|f'(x)| = \left|\frac{-1}{(x-3)^2}\right| = \frac{1}{(x-3)^2} < 1$$

Therefore $f'(x)$ is bounded on $(4,\infty)$ and hence by Theorem 19.6, f is uniformly continuous on $(4,\infty)$

19.6(a)

Notice that $f'(x) = \frac{1}{2\sqrt{x}}$, which is unbounded on $(0,1]$: If $M > 0$ large enough is given, let $x = \frac{1}{4M^2} \in (0,\infty)$ and therefore $f'(x) = \frac{1}{2\sqrt{\frac{1}{4M^2}}} = \frac{2M}{2} = M$ and hence $|f'(x)| \geq M$

However, since $f(x) = \sqrt{x}$ is continuous on $[0,1]$, which is closed and bounded, f is uniformly continuous on $[0,1]$, and hence uniformly continuous on $(0,1]$

This problem really shows that f' being bounded is just a sufficient condition, not a necessary condition!

19.6(b)

This follows since f is continuous on $[1, \infty)$, differentiable on $(1, \infty)$ and

$$|f'(x)| = \frac{1}{2\sqrt{x}} \leq \frac{1}{2}$$

Hence f' is bounded on $(1, \infty)$

Optional: Here's a more direct proof of this fact (good practice with the definition)

STEP 1: Scratchwork

$$\begin{aligned}
|f(x) - f(y)| &= \left|\sqrt{x} - \sqrt{y}\right| \\
&= \left|\left(\sqrt{x} - \sqrt{y}\right)\left(\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right)\right| \\
&= \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \\
&\leq 2\,|x - y| \;\text{ (Since } x, y \geq 1 \text{ and so } \sqrt{x}, \sqrt{y} \geq 1) \\
&< \epsilon
\end{aligned}$$

Which gives $|x - y| < 2\epsilon$

STEP 2: Actual Proof

Let $\epsilon > 0$ be given, let $\delta = 2\epsilon$, then if $x, y \in [1, \infty]$ and $|x - y| < \delta$, then $x, y \geq 1$ so $\sqrt{x}, \sqrt{y} \geq 1$ and so

$$\left|\sqrt{x} - \sqrt{y}\right| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{|x - y|}{2} < \frac{2\epsilon}{2} = \epsilon \checkmark \quad \square$$

19.8(a)

Let $f(x) = \sin(x)$. Then the MVT there is c between x and y such that

$$\begin{aligned} f(x) - f(y) &= f'(c)(x-y) \\ \Rightarrow |f(x) - f(y)| &= |f'(c)|\,|x-y| \\ \Rightarrow |\sin(x) - \sin(y)| &= \underbrace{|\cos(c)|}_{\leq 1}\,|x-y| \leq |x-y| \checkmark \end{aligned}$$

19.8(b)

Let $\epsilon > 0$ be given, let $\delta = \epsilon$, then if $|x-y| < \delta$, then by (a), we get

$$|\sin(x) - \sin(y)| \leq |x-y| < \epsilon \checkmark \quad \Box$$

20.11(a)

$$\lim_{x\to a} \frac{x^2 - a^2}{x-a} = \lim_{x\to a} \frac{(x-a)(x+a)}{x-a} = \lim_{x\to a} x + a = a + a = 2a$$

20.11(b)

$$\begin{aligned}\lim_{x\to b}\frac{\sqrt{x}-\sqrt{b}}{x-b} &= \lim_{x\to b}\frac{(\sqrt{x}-\sqrt{b})(\sqrt{x}+\sqrt{b})}{(x-b)(\sqrt{x}+\sqrt{b})}\\ &= \lim_{x\to b}\frac{x-b}{(x-b)(\sqrt{x}+\sqrt{b})}\\ &= \lim_{x\to b}\frac{1}{\sqrt{x}+\sqrt{b}}\\ &= \frac{1}{\sqrt{b}+\sqrt{b}}\\ &= \frac{1}{2\sqrt{b}}\end{aligned}$$

20.11(c)

$$\lim_{x\to a}\frac{x^3-a^3}{x-a} = \lim_{x\to a}\frac{(x-a)(x^2+ax+a^2)}{x-a} = \lim_{x\to a}x^2+ax+a^2 = a^2+a^2+a^2 = 3a^2$$

20.20(a)

Let $M > 0$ be given. Then since $\lim_{x\to a} f_2(x) = L_2$, with $\epsilon = 1$, there is $\delta_1 > 0$ such that if $0 < |x-a| < \delta_1$ then $|f_2(x) - L_2| < 1$, so in particular $f_2(x) - L_2 > -1$ so $f_2(x) > L_2 - 1$

Now since $\lim_{x\to a} f_1(x) = \infty$, there is $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$ then $f_1(x) > M - (L_2 - 1)$

Let $\delta = \min\{\delta_1, \delta_2\}$ then if $0 < |x-a| < \delta$, then $|x-a| < \delta_1$ and $|x_a| < \delta_2$ and so

$$(f_1+f_2)(x) = f_1(x) + f_2(x) > M - (L_2-1) + (L_2-1) = M\checkmark$$

Hence $\lim_{x\to a}(f_1 + f_2)(x) = \infty$ □

20.20(b)

Let $M > 0$ be given. Then since $\lim_{x\to a} f_2(x) = L_2$, with $\epsilon = \frac{L_2}{2} > 0$, there is $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$ then $|f_2(x) - L_2| < \frac{L_2}{2}$, so in particular $f_2(x) - L_2 > -\frac{L_2}{2}$ so $f_2(x) > L_2 - \frac{L_2}{2} = \frac{L_2}{2} > 0$

Now since $\lim_{x\to a} f_1(x) = \infty$, there is $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$ then $f_1(x) > \frac{M}{\frac{L_2}{2}}$

Let $\delta = \min\{\delta_1, \delta_2\}$ then if $0 < |x - a| < \delta$, then $|x - a| < \delta_1$ and $|x_a| < \delta_2$ and so

$$(f_1 f_2)(x) = f_1(x) f_2(x) > \frac{M}{\frac{L_2}{2}} \times \frac{L_2}{2} = M \checkmark$$

Hence $\lim_{x\to a}(f_1 f_2)(x) = \infty$ □