Additional Problem 1:

(a) Show that if E is a countable subset of $\mathbb{R}^d$ then E is measurable

(b) Find an uncountable subset of $\mathbb{R}$ that is measurable

(a) Denoted E as $E = \{P_1, P_2 \cdots P_k \cdots\} = \{P_k\}_{k=1}^{\infty}$, is a countable union of points $(P_1, P_2, \cdots, P_k, \cdots)$

Given $\epsilon > 0$, we may choose for each k an open set O_k with $E_k \subset O_k$ and $m_*(O_k) < \epsilon/2^k$ (Because E_k is a point in $\mathbb{R}^d$, such open set exists)

Then the union $O = \bigcup_{j=1}^{\infty} O_j$ is open (Because it is countable union of open sets)

$E \subset O$ and $m_*(O) \leq \sum_{j=1}^{\infty} m_*(O_j) < \epsilon$

Then $m_*(E) \leq m_*(O) = 0$ By Monotonically Observation

Then $m_*(E) = 0$

Then E is measurable By Property 2 of measurable sets

(b) $(0,1)$ is an measurable set because it's an open set

it is uncountable subset of $\mathbb{R}$ as well

Because: If it is countable

Any 0,1 sequence can 1-1 map to a number in $(0,1)$ by putting decimal point in front of it, like

$0.1010\cdots$

Because $(0,1)$ is countable, $\{0,1 \text{ sequence}\}$ is also countable.

However we know by diagonal argument, $\{0,1 \text{ sequence}\}$ is uncountable

Thus $(0,1)$ is an uncountable subset of $\mathbb{R}$.

More: Cantor Set:

It is the set of all numbers in $[0,1]$, which have a ternory expansion containing only digits 0 or 2 in first n spaces

2.44 The Cantor set The set which we are now going to construct shows that there exist perfect sets in R^1 which contain no segment.

Let E_0 be the interval $[0, 1]$. Remove the segment $(\frac{1}{3}, \frac{2}{3})$, and let E_1 be the union of the intervals

$$[0, \tfrac{1}{3}]\ [\tfrac{2}{3}, 1].$$

Remove the middle thirds of these intervals, and let E_2 be the union of the intervals

$$[0, \tfrac{1}{9}], [\tfrac{2}{9}, \tfrac{3}{9}], [\tfrac{6}{9}, \tfrac{7}{9}], [\tfrac{8}{9}, 1].$$

Continuing in this way, we obtain a sequence of compact sets E_n, such that

(a) $E_1 \supset E_2 \supset E_3 \supset \cdots$;
(b) E_n is the union of 2^n intervals, each of length 3^{-n}.

The set

$$P = \bigcap_{n=1}^{\infty} E_n$$

is called the *Cantor set*. P is clearly compact, and Theorem 2.36 shows that P is not empty.

If it is countable, we have elements in Cantor Set can be written out as:

$x^1 = 0.d_1^1 d_2^1 \cdots$ where $d_i^j \in \{0,2\}$
$x^2 = 0.d_1^2 d_2^2 \cdots$
$x^n = 0.d_1^n d_2^n \cdots$
$\vdots$

let $(d_1, d_2, \cdots d_n \cdots)$ be sequence that differs from the diagonal sequence $(d_1^1, d_2^2, d_3^3, \cdots)$ in every entry, s.t.

$$d_j = \begin{cases} 0 & \text{if } d_j^j = 2 \\ 2 & \text{if } d_j^j = 0 \end{cases}$$

Now $x = 0.d_1 d_2 \cdots d_n \cdots$ does not appear in the list, but $x \in$ Cantor Set,

which is a contradiction

$\Rightarrow$ Cantor Set is uncountable!

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Additional Problem 2: Let $E_1 = \mathcal{N}$ and $E_2 = \mathcal{N}^c$ (the complement is taken in $[0,1]$), where $\mathcal{N}$ is the non-measurable set from lecture, then although E_1 and E_2 are disjoint, prove that

$$m_\star(E_1 \cup E_2) \neq m_\star(E_1) + m_\star(E_2)$$

we have :

$m_*(E_1 \cup E_2) = m_*([0,1]) = 1$

$m_*(E_1) \neq 0$ Because $E_1 = N$ is non-measurable

By the properity of exterior measure, we know $m_*(E_1) > 0$

Then we consider $m_*(N^c)$

CLAIM: $m_*(N^c) = 1$

proof: Assume to contrary, We assume $m_*(N^c) = 1-2\varepsilon < 1$

Because $N^c \subset \mathbb{R}^d$, then $m_*(N^c) = \inf m_*(O)$, $N^c \subset O$, O is an open set.

Thus there exists an open set $U \subseteq [0,1]$ s.t. $N^c \subseteq O$ and $m_*(U) - m_*(N^c) < \varepsilon$. (By the definition of infimum)

thus $m_*(U) < m_*(N^c) + \varepsilon < 1-\varepsilon$

Because U is a measurable set, $[0,1]$ measurable, thus U^c measurable

Because $N^c \subset U$ then $U^c \subset N$

Let $\{r_k\}_{k=1}^\infty$ be an enumeration of all rationals in $[-1,1]$ and consider $\{U^c + r_k\}$: we have $U^c + r_k \subset N + r_k := N_k$, then

$\bigcup_k^\infty (U_c + r_k) \subset \bigcup_k N_k \subseteq [-1,2]$

$m(U_c) = m[0,1] - m(U) > \varepsilon$.

then $m\left(\bigcup_k^\infty (U_c + r_k)\right) = \infty$ which conflict with $m([-1,2]) = 3$

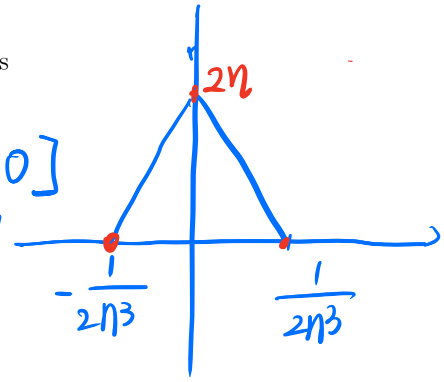
Thus $m_*(N^c) = 1$, proof of claim ends ※

Thus $m_*(E_1 \cup E_2) = 1 < 1 + m_*(E_1) = m_*(E_2) + m_*(E_1)$

Ends our proof ※

Additional Problem 3: Show that there is a non-negative continuous function f on $\mathbb{R}$ such that $\int_{\mathbb{R}} f(x)dx < \infty$ but $\limsup_{x\to\infty} f(x) = \infty$.

$$g_n(x) = \begin{cases} 2n + 4n^4 x & x \in [-\frac{1}{2n^3}, 0] \\ 2n - 4n^4 x & x \in (0, \frac{1}{2n^3}] \\ 0 & \text{otherwise.} \end{cases}$$



construction:

$$f(x) = \begin{cases} g_n\left(-\frac{1}{2n^3} + x - n\right) & [n, n+\frac{1}{n^3}], \ n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

1) nonnegative, because $g_n(x) \geq 0$

2) continuous, because $g_n(x)$ continuous and $g_n(-\frac{1}{2n^3}) = g_n(\frac{1}{2n^3}) = 0$

3) $\int_{\mathbb{R}} f(x)dx = \sum_{n=1}^{\infty} \int_{\mathbb{R}} g_n(x)dx = \sum_{n=1}^{\infty} (2n \cdot \frac{1}{n^3}) \times \frac{1}{2}$
$= \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

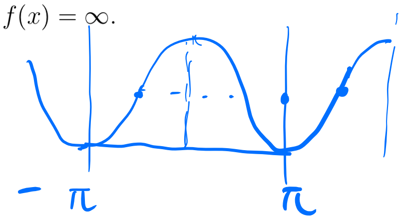
4) $\limsup_{x\to\infty} f(x) = \infty$

Because $f(n + \frac{1}{2n^2}) = g_n(0) = 2n \to \infty$ as $n \to \infty$ is a unbounded sequence.

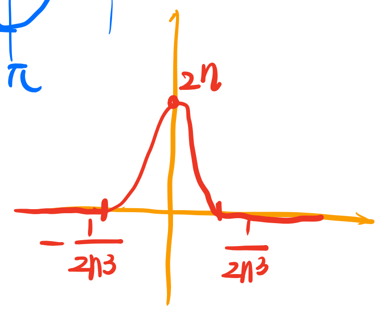
In next page, I construct a differentiable function f with same property.

Additional Problem 3: Show that there is a non-negative continuous function f on $\mathbb{R}$ such that $\int_{\mathbb{R}} f(x)dx < \infty$ but $\limsup_{x\to\infty} f(x) = \infty$.

$$\int_{-\pi}^{\pi} (\cos x + 1)\,dx = 2\pi$$



$$g_n(x) = n\left(\cos(2n^3\pi x) + 1\right)\cdot \mathbb{1}_{\left[-\frac{1}{2n^3}, \frac{1}{2n^3}\right]} \Rightarrow$$



Then $\int_{\mathbb{R}} g_n(x) = \int_{-\frac{1}{2n^3}}^{\frac{1}{2n^3}} n\left(\cos(2n^3\pi x)+1\right)dx$

$$= n\cdot\frac{1}{n^3} + n\cdot\int_{-\frac{1}{2n^3}}^{\frac{1}{2n^3}} \cos(2n^3\pi x)\,dx$$

$$= \frac{1}{n^2} + n\cdot\frac{1}{2n^3\pi}\cdot\underbrace{\int_{-\frac{1}{2n^3}}^{\frac{1}{2n^3}} \cos(2n^3\pi x)\,d(2n^3\pi x)}_{=0}$$

$$= \frac{1}{n^2}$$

$\max_x g_n(x) = g_n(0) = 2n$ $\qquad$ $f(n) = g_n\left(-\frac{1}{2n^3}\right)$

construction:

$$f(x) = \begin{cases} g_n\left(-\frac{1}{2n^3} + x - n\right) & \left[n, n+\frac{1}{n^3}\right],\ n\in\mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

0) non-negative because $g_n(x) \geq 0$ for all $x\in\mathbb{R}$

1) continuous $f'(n) = g'\left(-\frac{1}{2n^3}\right) = n(\sin\pi) = 0 = n(-\sin\pi) = g'\left(\frac{1}{2n^3}\right) = f'\left(n+\frac{1}{n^3}\right)$

it is actually differentiable! and thus continuous

2) $\int_{\mathbb{R}} f(x)dx = \sum_{n=1}^{\infty} \int_{\mathbb{R}} g_n(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$

3) $\limsup_{x\to\infty} f(x) = \infty$ because we can find a unbounded sequence:

$f\left(n+\frac{1}{2n^2}\right) = g_n(0) = 2n \to \infty$ as $n\to\infty$

Thus f is the function we want.

Additional Problem 4: Consider

$$f_a(x) = \begin{cases} \frac{1}{|x|^a} & \text{if } |x| \le 1 \\ 0 & \text{if } |x| > 1 \end{cases} \quad \text{and} \quad g_a(x) = \begin{cases} 0 & \text{if } |x| \le 1 \\ \frac{1}{|x|^a} & \text{if } |x| > 1 \end{cases}$$

Show that f_a is integrable precisely when $a < d$ and g_a is integrable precisely when $a > d$ (see hints). Integrable means that $\int |f_a| < \infty$.

Additional Problem 4: Here is is useful to use the polar coordinates formula (no need to prove)

$$\int_{\mathbb{R}^d} f(x)dx = \int_0^\infty \left(\int_{|x|=r} f(x)dS(x) \right) dr$$

Here $S(x)$ is the "surface measure" on the sphere. You're also allowed to assume that the surface measure of $\{|x| = r\}$ is $C(d)\, r^{d-1}$ where C is some constant (depending on d)

$\boxed{\text{lemma}}$: $\int_0^1 \frac{1}{r^p} dr$ diverges when $p \geq 1$, converges when $p < 1$

$\int_1^\infty \frac{1}{r^p} dr$ diverges when $p \leq 1$, converges when $p > 1$

proved afterwards

$$\int_{\mathbb{R}^d} |f_a(x)| dx = \int_0^\infty \left(\int_{|x|=r} f(x)\, dS(x) \right) dr$$
$$= \int_0^1 \left(\int_{|x|=r} \frac{1}{r^a}\, dS(x) \right) dr$$
$$= \int_0^1 \left(\frac{1}{r^a}\, C(d)\, r^{d-1} \right) dr$$
$$= C(d) \int_0^1 \frac{r^{d-1}}{r^a} dr$$
$$= C(d) \int_0^1 \frac{1}{r^{a-d+1}} dr$$

, is integrable only when $a-d<0$ i.e. $a<d$

$$\int_{\mathbb{R}^d} |g_a(x)| dx = \int_1^\infty \left(\int_{|x|=r} \frac{1}{r^a}\, dS(x) \right) dr$$
$$= C(d) \cdot \int_1^\infty \frac{1}{r^{a-d+1}} dr$$

is integrable only when $a-d>0$, i.e. $a>d$

which ends the proof ※

Proof of lemma:

for: $\int_0^1 \frac{dx}{x^p}$,

- $p>1$: $\int_0^1 \frac{dx}{x^p}$ (improper Riemann) $= \lim_{\varepsilon\to 0} \int_\varepsilon^1 \frac{dx}{x^p}$ (Lebesgue) $= \lim_{\varepsilon \to 0} \int_\varepsilon^1 \frac{dx}{x^p} = -\frac{1}{p-1} + \lim_{\varepsilon\to 0} \frac{1}{(p-1)\varepsilon^{p-1}} = +\infty$
- $p=1$ clearly diverges ($[\log x]_0^1$)
- $p<1$ converges because $\lim_{\varepsilon\to 0} \frac{1}{(p-1)\varepsilon^{p-1}} = \lim_{\varepsilon\to 0} \frac{\varepsilon^{1-p}}{p-1} = 0$

for $\int_1^\infty \frac{dx}{x^p} = \int_0^1 \frac{1}{x^{2-p}}\, dx$

Additional Problem 5: Show that if $\int_E f(x)dx = 0$ for every measurable E then $f(x) = 0$ a.e.

Assume to the contrary: we assume that $f(x)=0$ a.e. not true

let $m(\{x: f(x)\neq 0\})>0$

Then we have $\{x: f(x)\neq 0\} = \{x: f(x)>0\} \cup \{x: f(x)<0\}$

Then at least one of two sets have positive measure.

WLOG we assume $m(\{f(x)>0\})>0$

$$\{x: f(x)>0\} = \bigcup_K \{x: f(x)>\frac{1}{K}\} = \bigcup_K E_K$$

then at least one of E_K has positive measure

Say E_m has positive measure

$$\int_{E_m} f(x)\,dx \geq \int_{E_m} \cdot \frac{1}{m} = \frac{1}{m} m(E_m) > 0$$

which leads to contradiction.

So $f(x)=0$ a.e. which ends the proof.