MATH S4062 – HOMEWORK 9

Additional Problem 1:

(a) Show that if E is a countable subset of $\mathbb{R}^d$ then E is measurable

(b) Find an uncountable subset of $\mathbb{R}$ of measure 0 that is measurable

Additional Problem 2: Let $E_1 = \mathcal{N}$ and $E_2 = \mathcal{N}^c$ (the complement is taken in $[0,1]$), where $\mathcal{N}$ is the non-measurable set from lecture, then although E_1 and E_2 are disjoint, prove that

$$m_\star(E_1 \cup E_2) \neq m_\star(E_1) + m_\star(E_2)$$

Additional Problem 3: Show that there is a non-negative continuous function f on $\mathbb{R}$ such that $\int_{\mathbb{R}} f(x)dx < \infty$ but $\limsup_{x\to\infty} f(x) = \infty$.

Additional Problem 4: Consider

$$f_a(x) = \begin{cases} \frac{1}{|x|^a} & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases} \quad \text{and} \quad g_a(x) = \begin{cases} 0 & \text{if } |x| \leq 1 \\ \frac{1}{|x|^a} & \text{if } |x| > 1 \end{cases}$$

Show that f_a is integrable precisely when $a < d$ and g_a is integrable precisely when $a > d$ (see hints). Integrable means that $\int |f_a| < \infty$.

Additional Problem 5: Show that if $\int_E f(x)dx = 0$ for every measurable E then $f(x) = 0$ a.e.

Date: Due: Wednesday, August 10, 2022.

Hints:

Additional Problem 1: For (a) use a $\frac{\epsilon}{2^n}$ argument and for (b), consider the Cantor set

Additional Problem 2: The main goal is showing that $m_\star(\mathcal{N}^c) = 1$. To do this, suppose by contradiction that $m_\star(\mathcal{N}^c) < 1$. Then there is a measurable set $U \subseteq [0,1]$ such that $\mathcal{N}^c \subseteq U$ and $m_\star(U) < 1 - \epsilon$ (no need to prove this). Then consider the translates $(U^c + r_k) \subseteq \mathcal{N}_k$ (notation as in lecture) and hence $\bigcup_k (U^c + r_k) \subseteq \bigcup_k \mathcal{N}_k \subseteq [-1, 2]$

Additional Problem 3: Consider the continuous version of the function equal to n on the segment $\left[n, n + \frac{1}{n^3}\right)$ for $n \geq 1$. Of course you can integrate the function you usually would in terms of areas

Additional Problem 4: Here is is useful to use the polar coordinates formula (no need to prove)

$$\int_{\mathbb{R}^d} f(x)dx = \int_0^\infty \left(\int_{|x|=r} f(x) dS(x) \right) dr$$

Here $S(x)$ is the "surface measure" on the sphere. You're also allowed to assume that the surface measure of $\{|x| = r\}$ is $C(d)\, r^{d-1}$ where C is some constant (depending on d)

Additional Problem 5: For every $k \geq 1$, consider $E_k = \left\{ x \in E \mid f(x) \geq \frac{1}{k} \right\}$ and notice that $f(x) \geq \left(\frac{1}{k}\right) 1_{E_k}(x)$ and write $\{f > 0\}$ in terms of E_k