LEBESGUE INTEGRAL

1. Lebesgue Integral

LEVEL 1: (f simple) The Lebesgue Integral of $\phi = \sum_{k=1}^{N} a_k \, 1_{E_k}$ is

$$\int_{\mathbb{R}^d} \phi(x)dx = \sum_{k=1}^{N} a_k m(E_k)$$

LEVEL 2: (f bounded and finite support)

$$\int f(x)dx = \lim_{k\to\infty} \int \phi_k(x)dx$$

Where $\{\phi_k\}$ is any sequence of bounded simple functions (with same support as f) such that $\phi_k \to f$ pointwise

LEVEL 3: ($f \geq 0$)

$$\int f(x)dx = \sup_g \int g(x)dx$$

Where the sup is taken over all g from **LEVEL 2** with $0 \leq g \leq f$, that is g is bounded and supported on a set of finite measure.

LEVEL 4: (General f)

$$\int f = \int f^+ - \int f^-$$

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Where $f = f^+ - f^-$ with $f^+ = \max(f(x), 0)$ and $f^- = \max(-f(x), 0)$

2. Convergence Theorems

Bounded Convergence Theorem: If $\{f_n\}$ is a sequence of functions bounded by M (and supported on E with $m(E) < \infty$) and $f_n \to f$ a.e., then

$$\lim_{n\to\infty} \int |f_n - f| = 0$$

Fatou's Lemma If $f_n \geq 0$ and $f_n \to f$ pointwise a.e. then

$$\int f \leq \liminf_{n\to\infty} \int f_n$$

Monotone Convergence Theorem: If $f_n \geq 0$ with $f_n \nearrow f$ then

$$\lim_{n\to\infty} \int f_n = \int f$$

Dominated Convergence Theorem: Suppose $\{f_n\}$ is a sequence such that $f_n \to f$ a.e. and $|f_n| \leq g$ where g is an integrable function

$$\text{Then } \lim_{n\to\infty} \int |f_n - f| = 0$$

3. Videos

- Integration Sucks
- Lebesgue Integral Overview
- Lebesgue Integral Example
- Riemann vs Lebesgue Integral

- $\int_0^1 x^2 dx$ four ways
- Dominated Convergence Theorem