LECTURE 1: NATURAL AND RATIONAL NUMBERS

1. Introduction

Hello everyone and welcome to Math 409, fun with epsilon! My name is Peyam and I'll be your instructor this semester

- **Logistics:** All the info is on the syllabus, which can be found on Howdy
- **Office Hours:** W 3−4:30 pm and Th 2:15−3:45 pm via Zoom. Please come, I'd be happy to help! You are allowed to request in person office hours.
- **Textbook:** *Elementary Analysis: The Theory of Calculus (2nd edition)* by Ross. This book is freely available through Springerlink. There are also some excellent book recommendations on the syllabus.
- **Resources:** Here are some resources you can use:
  - Course Website: For the lecture notes, YouTube videos, and practice exams. NOT set up yet, so in the meantime everything is posted on Canvas
  - Canvas Check your grades and submit the homework
  - Campuswire A really cool forum; use it to ask questions
  - YouTube Channel My YouTube channel, for videos related to this course

Date: Tuesday, August 31, 2021.

- TikTok Channel For other fun videos

- **Grading:**
  - **HW 25 %**, due every Friday by 11:55 pm on Canvas, with the exception of Exam Weeks and Thanksgiving. The first HW is due this Friday; the lowest 3 homework assignments are dropped
  - **Midterms 20 % each**. 2 Midterms, during class, on the following days:

    Midterm 1: Thursday, Sep 30

    Midterm 2: Thursday, Nov 4

  - **Final 35 %** Wednesday, Dec 15, 8-10 am, cumulative
  - **Extra Credit 1 %**: Given to the top posters on Campuswire

- **Grades:** You will be graded according to the scale in the syllabus, so everyone can get an A if they work hard. I will try my best to be as generous as I can

- **Finally:** Sit back, relax, and enjoy the show. Teaching this course is literally a dream come true for me and I hope you'll enjoy it as much as I do. There will be lots of suffering, but I promise you that it'll be worth it. This is the course that will make you understand and enjoy math ☺

2. What is $\mathbb{N}$?

How does one start an analysis course? It's like starting the universe; there are so many different ways of doing it! But let's begin with the basic unit of analysis: the natural numbers.

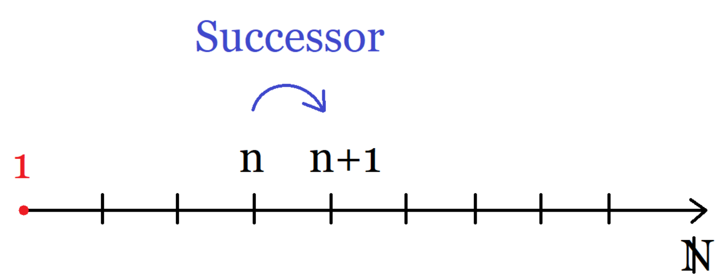
Video: What is a number?

Notation: $\mathbb{N} = \{1, 2, 3, \cdots\}$

This is the definition that you've known your whole life... except it makes no sense! What even *is* a number? And what makes $\mathbb{N}$ different from the integers $\mathbb{Z}$ or the real numbers $\mathbb{R}$? There are two special features of $\mathbb{N}$:

(1) $\mathbb{N}$ has a smallest element, called 1

(2) Every integer n has a successor $n+1$, as in the following picture:



But then how do you *define* $\mathbb{N}$? I'd like to mention that there's an explicit way of constructing $\mathbb{N}$ (see HW), but in this course we take an axiomatic approach. An axiom is like a rule to a game; math is the art of combining axioms to get meaningful statements.

Peano's Axioms:

There is a set $\mathbb{N}$ such that:

(1) 1 is in $\mathbb{N}$

(2) If $n \in \mathbb{N}$, then its successor $n+1 \in \mathbb{N}$

(3) 1 is not the successor of any element in $\mathbb{N}$ (see picture above)

(4) $n \neq m \Rightarrow n+1 \neq m+1$ (different numbers have different successors)

(5) Induction Axiom (see below)

Those axioms should hopefully seem reasonable to you. Using them, we can define $2 = 1+1$ (successor of 1), $3 = 2+1$ (successor of 2) and so on. Can we reach *all* numbers in that way, by using successors? The next axiom says yes:

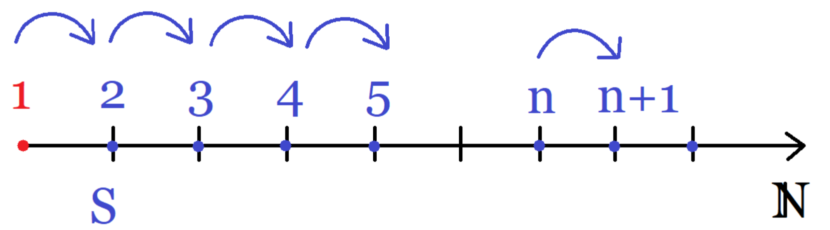
"Induction" Axiom:

Suppose S is a subset of $\mathbb{N}$ such that:

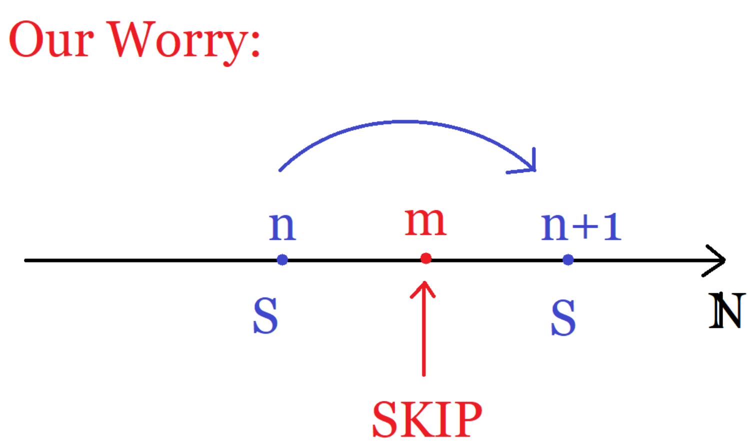
(1) $1 \in S$

(2) If $n \in S$, then $n+1 \in S$

Then $S = \mathbb{N}$



What is this saying? Since 1 is in S, its successor $1 + 1 = 2$ is also in S, hence $2 + 1 = 3$ is also in S, and hence 4 is in S and so forth. *In theory*, it's possible that we're skipping natural numbers that way, but (5) says that it is not so; S must indeed be all of $\mathbb{N}$.



This is certainly not true for $\mathbb{R}$: If you jump from 1 to 2, then you're skipping lots of real numbers like $\frac{3}{2}, \sqrt{2}, \frac{\pi}{2}$

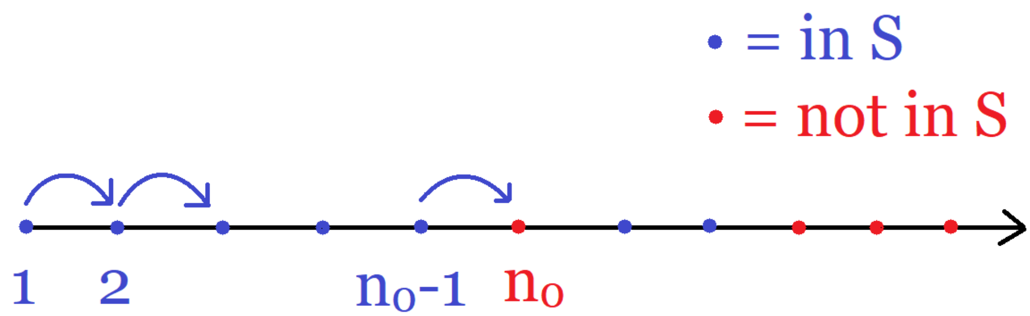
Note: Below I'll explain why it's called an induction axiom. Also, in the optional appendix (at the end), you can see a small proof that puts (5) in action.

3. Mini Analysis Proof

To give you a taste of how a typical analysis proof works, let me give you a "proof" as to why (5) is true. I would like to emphasize that it is **not** an actual proof, see below as to why it isn't.

"Proof" of (5)**:** Suppose (5) were false, that is there is a subset S of $\mathbb{N}$ that satisfies (1) and (2), but $S \neq \mathbb{N}$.

Let n_0 be the first/smallest number that is not in S:



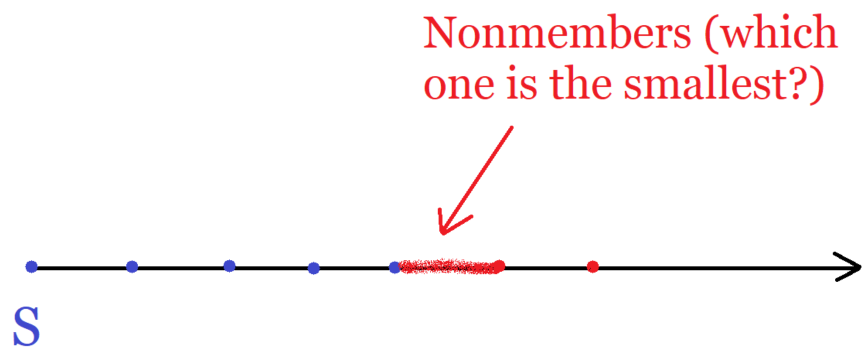
Notice $n_0 \neq 1$, because $1 \in S$ but $n_0 \notin S$

Moreover, since n_0 is the smallest nonmember of S, we must have $n_0 - 1 \in S$ (otherwise $n_0 - 1$ would be an even smaller nonmember)

But since $n_0 - 1 \in S$, then by (2), $(n_0 - 1) + 1 = n_0 \in S$, which contradicts $n_0 \notin S \Rightarrow\Leftarrow$ $\square$

Note: This is not a proof because, first of all, what does $n_0 - 1$ even mean? We don't know that n_0 is the successor of a number.

More importantly, even though there are numbers not in S, how do we know there is a *smallest* such number? We could have a situation like follows where there's a continuum of numbers not in S:



This will be extremely important in Section 4.

4. Induction

Video: What is Induction?

(5) is called the Induction Axiom is because it is the basis of:

Mathematical Induction:

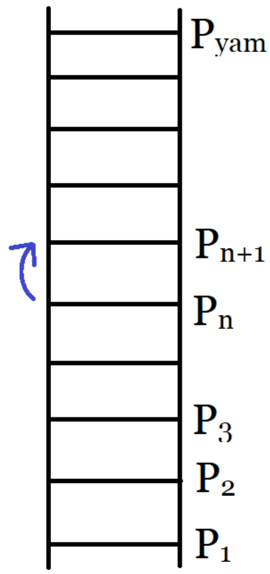
Let P_n be a proposition and suppose

(1) P_1 is true

(2) For all $n \in \mathbb{N}$, if P_n is true, then P_{n+1} is true

Then P_n is true for all $n \in \mathbb{N}$

(Notice how similar this is to (5))



Example 1:

Show by induction on n that if $r \neq 1$ and $n \geq 1$, then:

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

Advice: On your homework and exams, it's very important to be as thorough as possible Everything I say below is important. Do **not** skip steps in your proof, even those that are "obvious" to you! If in doubt, write it out.

Let P_n be the proposition:

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

Base Case: Show P_1 is true, that is:

$$1 + r = \frac{1 - r^2}{1 - r}$$

But

$$\frac{1 - r^2}{1 - r} = \frac{(1 - r)(1 + r)}{1 - r} = 1 + r \checkmark$$

Inductive Step: Suppose P_n is true, that is:

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

Show P_{n+1} is true, that is:

$$1 + r + r^2 + \cdots + r^{n+1} = \frac{1 - r^{n+2}}{1 - r}$$

But:

$$\begin{aligned}
1 + r + r^2 + \cdots + r^{n+1} &= (1 + r + \cdots + r^n) + r^{n+1} \\
&= \frac{1 - r^{n+1}}{1 - r} + r^{n+1} \text{ By the } \textbf{inductive hypothesis} \\
&= \frac{1 - r^{n+1} + r^{n+1}(1 - r)}{1 - r} \\
&= \frac{1 - \cancel{r^{n+1}} + \cancel{r^{n+1}} - r^{n+2}}{1 - r} \\
&= \frac{1 - r^{n+2}}{1 - r}
\end{aligned}$$

Therefore P_{n+1} is true, and hence P_n is true, that is, for all $n \in \mathbb{N}$,

$$1 + r + r^2 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

5. Triangle Inequality

Video: Induction Example

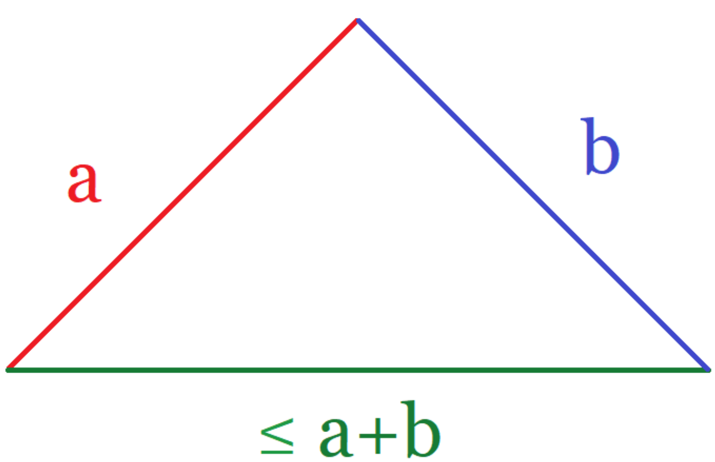
Analysis not only deals with numbers, but also with functions, so let's do one more induction example, but with functions. For this we'll need one of the most important inequalities of this course:

IMPORTANT: Triangle Inequality:

If a and b are real numbers, then

$$|a+b| \leq |a| + |b|$$

Interpretation: The third leg of a triangle is always smaller than (or equal to) the sum of the other two legs.

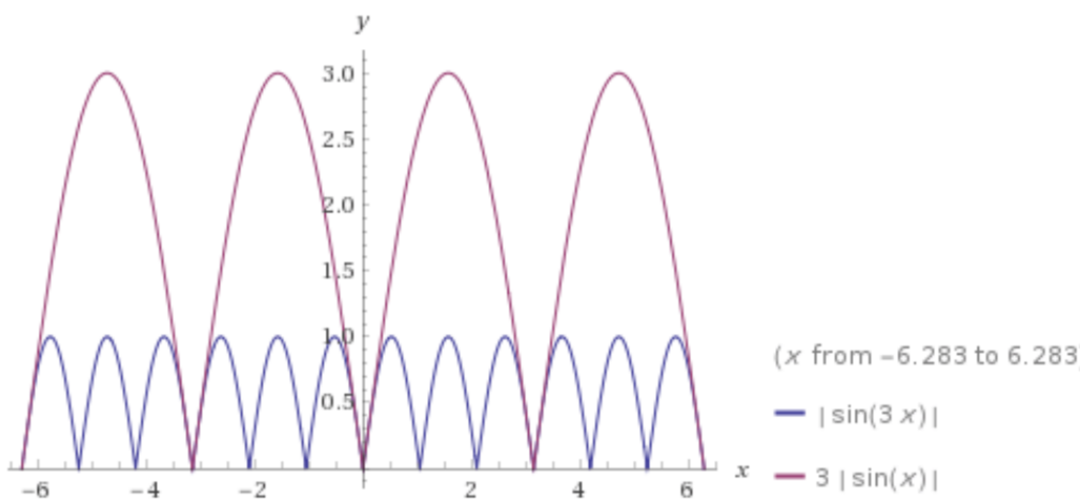


Example 2:

Show that for all $n \in \mathbb{N}$ and all $x \in \mathbb{R}$, we have

$$|\sin(nx)| \leq n\,|\sin(x)|$$

Picture: The picture illustrates the case $n = 3$ (Courtesy Wolfram Alpha)



Let P_n be the proposition: For all $x \in \mathbb{R}$, $|\sin(nx)| \leq n\,|\sin(x)|$

Base Case: For $n = 1$, we have $|\sin(1x)| \leq 1\,|\sin(x)|$ ✓.

Inductive Step: Suppose P_n is true, that is $|\sin(nx)| \leq n|\sin(x)|$.

Show P_{n+1} is true, that is: $|\sin((n+1)x)| \leq (n+1)|\sin(x)|$. But:

$$\begin{aligned}
|\sin((n+1)x)| &= |\sin(nx+x)| \\
&= |\sin(nx)\cos(x) + \cos(nx)\sin(x)| \\
&\quad \text{Here we used } \sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B) \\
&\leq |\sin(nx)| \underbrace{|\cos(x)|}_{\leq 1} + \underbrace{|\cos(nx)|}_{\leq 1} |\sin(x)| \\
&\quad \text{(By the Triangle Inequality)} \\
&\leq |\sin(nx)| + |\sin(x)| \\
&\leq n|\sin(x)| + |\sin(x)| \qquad \text{(By the } \textbf{inductive hypothesis}\text{)} \\
&= (n+1)|\sin(x)|
\end{aligned}$$

Therefore P_{n+1} is true, and hence P_n is true for all n, that is $|\sin(nx)| \leq n|\sin(x)|$ for all x. $\square$

6. (Ir)rational Numbers

Video: $\sqrt{2}$ is irrational

Now that we we've learned about natural numbers, let's talk about rational numbers!

Definition:

We say x is **rational** (and we write $x \in \mathbb{Q}$) if $x = \frac{p}{q}$ where p and q are integers with $q \neq 0$

Not every number is rational and in fact the following is one of the most classical theorems in all of math:

Theorem:

$\sqrt{2}$ is irrational

Proof: Suppose $\sqrt{2} = \frac{p}{q}$, where p and $q \neq 0$ are integers.

Without loss of generality (WLOG), assume that p and q have no factors in common

(Why? Say p and q have a factor of 3 in common, then $p = 3k$ and $q = 3l$. But then $\frac{p}{q} = \frac{3k}{3l} = \frac{k}{l}$ and you can just repeat the same proof with k and l instead of p and q)

Squaring both sides of $\sqrt{2} = \frac{p}{q}$, you get $\frac{p^2}{q^2} = 2$, so $p^2 = 2q^2$.

But then p^2 is even, so p is even (Why? The contrapositive statement is: If p is odd, then p^2 is odd, which you can show using the definition of odd[1]), so $p = 2m$ for some integer m

But then:

$$p^2 = 2q^2 \Rightarrow (2m)^2 = 2q^2 \Rightarrow 4m^2 = 2q^2 \Rightarrow q^2 = 2m^2$$

But then q^2 is even, so q is even, so $q = 2n$ for some integer n.

But then p and q have a factor of 2 in common, which contradicts WLOG. $\Rightarrow\Leftarrow$ $\square$

[1] If p is odd, then $p = 2k+1$, so $p^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2n + 1$ for $n = 2k^2 + 2k \in \mathbb{Z}$

Note: Essentially, this is saying that, if $\sqrt{2}$ were a fraction, then it would be a fraction with infinitely many factors of 2 that would cancel out, like

$$\sqrt{2} = \frac{\cancel{2}\cancel{2}\cancel{2}\cdots}{\cancel{2}\cancel{2}\cancel{2}\cdots}$$

Which, strictly speaking, makes no sense. Fractions should have an end!

Remarks:

(1) There are other kinds of irrational numbers, like $\sqrt{3}$, $\sqrt[3]{2}$, $\log_2(3)$, but also e and π. In case you're curious, feel free to check out the following optional videos: e is irrational and π is irrational

(2) Since $\mathbb{Q}$ is countable but $\mathbb{R}$ is uncountable (see this optional video as to why), there are **many** more irrational numbers than rational numbers. In fact, if you pick a real number at random, the probability that it's irrational is 1! (WOW!)

(3) In case you're wondering how to actually construct $\mathbb{Q}$, there's a very elegant way of constructing rational numbers from integers, which you can find on the next homework or in this video: What is $\mathbb{Q}$?

7. Algebraic Numbers

Video: Algebraic Numbers

Although $\sqrt{2}$ is irrational, it's not very bad because it is a zero of a polynomial with integer coefficients, namely $x^2 - 2$. We call such kind of numbers **algebraic**:

Definition:

A real number x is called **algebraic** if it is the root of a polynomial with integers coefficients. That is, there exist integers $a_n, a_{n-1}, \cdots, a_1, a_0$ (with $a_n \neq 0$) such that

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

Example:

Is $x = \sqrt[3]{\frac{\sqrt{2}-3}{5}}$ algebraic? (It sure looks horrible, doesn't it!)

$$\begin{aligned} x =& \sqrt[3]{\frac{\sqrt{2}-3}{5}} \\ \Rightarrow x^3 =& \frac{\sqrt{2}-3}{5} \\ \Rightarrow 5x^3 =& \sqrt{2} - 3 \\ \Rightarrow \sqrt{2} =& 5x^3 + 3 \\ \Rightarrow 2 =& (5x^3+3)^2 \\ \Rightarrow 2 =& 25x^6 + 30x^3 + 9 \\ \Rightarrow& 25x^6 + 30x^3 + 7 = 0 \end{aligned}$$

So **YES** ("It's not a bad number")

Definition:

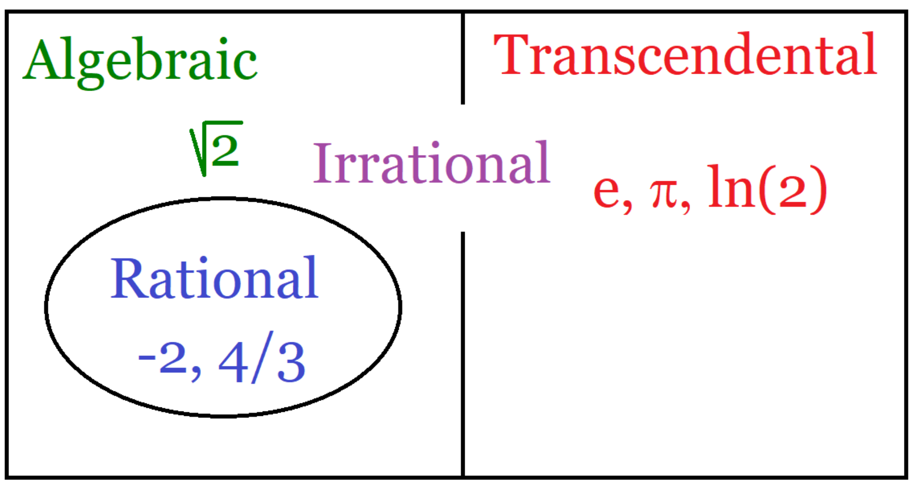
If x is not algebraic, then x called **transcendental**

(Like your "favorite" Calculus book[2] Stewart's *Calculus, Early Transcendentals*)

The two most famous transcendental numbers are e and π, see the following **optional** videos for proofs: e is transcendental and π is transcendental)

Note: Again, you can show that the algebraic numbers are countable (see the next homework or this video), so because $\mathbb{R}$ is uncountable, there are many more transcendental numbers than algebraic ones.

The following picture summarizes the dichotomy between rational vs. irrational numbers, and algebraic vs. transcendental numbers.



[2]Not!

8. Optional Appendix: Axiomatic Proof

In case you're curious about how to apply (5) in math, here's a statement that's hopefully obvious to you, but not as obvious to show:

Claim

For all $n \in \mathbb{N}$, $n + 1 \neq n$

Proof: Let S be the set of $n \in \mathbb{N}$ such that $n + 1 \neq n$.

First of all, $1 \in S$, because even though $1 + 1$ is a successor of an element of $\mathbb{N}$ (namely of 1), 1 is not a successor of any element in $\mathbb{N}$ by Axiom (3). Hence $1 + 1 \neq 1$, so by definition of S, $1 \in S$ ✓

Now suppose $n \in S$, that is $n \neq n + 1$, but then by Axiom (4), $n+1 \neq n+1+1$, so $(n+1) \neq (n+1)+1$ so by definition of S, we have $n + 1 \in S$ ✓

So by (5), we have $S = \mathbb{N}$, that is for **all** $n \in \mathbb{N}$, we have $n+1 \neq n$ $\square$