LECTURE 1: SEQUENCES OF FUNCTIONS

1. Introduction

Hello everyone and welcome to Math 4062, an exciting Analysis adventure awaits you! My name is Peyam and I'll be your instructor this summer.

- **Logistics:** All the info is on the syllabus, which can be found on the Course Website
- **Lectures:** MTWTh 10:45-12:20 pm in 312 Math Building. Attendance is optional, although I highly recommend coming to lecture
- **Office Hours:** MTWTh 12:30-1:15 pm in 600 Math Building. Please come, I'd be happy to help
- **Textbook:** Our official textbook is *Principles of Mathematical Analysis* by Rudin. I'm not always going to follow the textbook, so I highly recommend reading the lecture notes.
- **Resources:** Here are some resources you can use:
  - Course Website Lecture notes, homework, and study material.
  - Canvas Submit the homework and check your grades
  - Piazza A really cool forum; use it to ask questions
  - YouTube Channel My YouTube channel, some videos will be related to this course

Date: Tuesday, July 5, 2022.

- **Grading:**
  - **HW 35%** due every TuTh by 11:59 pm on Canvas, with the exception of exam days. The first HW is due this Thursday; the lowest homework is dropped. There will be quite a bit of homework, but remember that it's also worth a lot of your grade.
  - **Midterm 25%** One in-class midterm on **Tuesday, July 26**. Exams are closed book and closed notes
  - **Final 40%** One in-class final on **Thursday, August 11**, cumulative
  - **Extra Credit 1%** Given to the 10 top posters on Piazza
- **Grades:** You will be graded according to the scale in the syllabus, so everyone can get an A if they work hard. I will curve the grades if they are too low, and try my best to be as generous as I can
- **Finally:** Sit back, relax, and enjoy the show. Teaching this course is literally a dream come true for me and I hope you'll enjoy it as much as I do. There will be lots of suffering, but I promise you that it'll be worth it. ☺

2. Sequences of Functions

Previously: (in Analysis I) You learned what it means for a *sequence* (s_n) to converge to s:

Definition: $s_n \to s$ if for all $\epsilon > 0$ there is N such that for all n if $n > N$ then $|s_n - s| < \epsilon$.

That is, eventually all the terms of (s_n) are as close to s as you want.

Our goal in this chapter is to generalize this to functions: what does it mean for a sequence of functions (f_n) to converge to f?

The simplest idea is just to consider convergence at every point:

Definition: $f_n \to f$ **pointwise** if, for every x, we have

$$\lim_{n\to\infty} f_n(x) = f(x)$$

Example: Let $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(x) = x^n$. What function does f_n converge to pointwise?

If $0 \leq x < 1$, then $\lim_{n\to\infty} x^n = 0$

If $x = 1$ then $f_n(1) = 1^n = 1 \to 1$

Therefore, f_n converges pointwise to:

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Notice: Even though each f_n is continuous on $[0,1]$, the limit function f fails to be continuous!

So although the notion of pointwise convergence is intuitive, it's bad in the sense that many properties such as continuity or differentiability are lost under pointwise convergence.

What we need is a notion called **uniform convergence** which is harder to understand at first, but much more powerful.

3. Uniform Convergence

Uniform here means "independent," the convergence is independent of x (the same for all x)

Definition: $f_n \to f$ **uniformly** if for all $\epsilon > 0$ there is N such that **for all** x, if $n > N$, then we have

$$|f_n(x) - f(x)| < \epsilon$$

The point is that here the N is independent of x, it works for all x.

Graphically: Notice

$$|f_n - f| < \epsilon \Rightarrow f - \epsilon < f_n < f + \epsilon$$

So for all large n, the graph of f_n is contained in an ϵ-tube around the graph of f. This is not the case for non-uniform convergence. (see pictures in lecture)

Example: The sequence $f_n : [0, 2] \to \mathbb{R}$ defined by the piecewise linear function connecting $(0, 0), (1, \frac{1}{n}), (2, 0)$ converges uniformly to $f(x) = 0$. No matter how small, every $\epsilon-$tube around f eventually contains all f_n for n large enough.

Let's now see how awesome this concept is! More precisely, we'll see that many properties of functions (such as continuity and differentiability) is preserved under uniform convergence

4. Uniform Convergence and Continuity

Theorem: (Continuity) If $f_n \to f$ uniformly and each f_n is continuous at x_0, then f is continuous at x_0.

Proof:[1] This is a typical $\frac{\epsilon}{3}$ proof:

Let $\epsilon > 0$ and x_0 be given. We need to find $\delta > 0$ such that for all x, if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$.

STEP 1: Since $f_n \to f$ uniformly, there is N such that for all $n \geq N$ and all x, we have

$$|f_n(x) - f(x)| < \frac{\epsilon}{3}$$

STEP 2: Since f_N is continuous at x_0 there is $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|f_N(x) - f_N(x_0)| < \frac{\epsilon}{3}$$

STEP 3: With that δ, if $|x - x_0| < \delta$, we get

$$\begin{aligned}
|f(x) - f(x_0)| =& |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)| \\
\leq& |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\
<& \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
=& \epsilon \checkmark
\end{aligned}$$

Here we used uniform convergence, continuity of f_N, and uniform convergence again, □

Note: Uniform convergence is essential here, as the x^n example shows. Even worse, sometimes the limit function f is *nowhere* continuous (see

[1] This proof is taken from Pugh's book, Chapter 4 Theorem 1

Example 7.4 in Rudin)

5. Uniform Convergence and Integration

Theorem: If $f_n \to f$ uniformly and each f_n is Riemann integrable on $[a, b]$, then so is f, and

$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$$

In other words, $f_n \to f \Rightarrow \int f_n \to \int f$ we can integrate the uniform convergence without feeling guilty.

Note: This is very useful in practice: f is usually a complicated function (think e^{x^2}), but which can be approximated by simpler functions f_n (think polynomials). This says that to calculate $\int_a^b f(x)dx$ (hard), you just take the limit of the approximate integrals $\int_a^b f_n(x)dx$ (easy)

Proof: Recall that integrable just means that $\underline{\int} f dx = \overline{\int} f dx$

STEP 1: Define ϵ_n as:

$$\epsilon_n = \sup \{|f_n(x) - f(x)| \mid x \in [a, b]\}$$

Intuitively, ϵ_n is the biggest possible spread between f_n and f (see picture in lecture)

Since $f_n \to f$ uniformly, we have $\lim_{n\to\infty} \epsilon_n = 0$ (we will discuss this fact in detail later)

STEP 2: By definition of ϵ_n we have

$$|f_n - f| \leq \epsilon_n \Rightarrow f_n - \epsilon_n \leq f \leq f_n + \epsilon_n$$

Lower and upper integrating on $[a, b]$, we get

$$\underline{\int} f_n - \epsilon_n dx \leq \underline{\int} f dx \text{ and } \overline{\int} f dx \leq \overline{\int} f_n + \epsilon_n dx$$

But since $f_n \pm \epsilon_n$ are integrable, we can remove the bar and get

$$\int_a^b f_n - \epsilon_n \leq \underline{\int} f dx \leq \overline{\int} f dx \leq \int_a^b f_n + \epsilon_n dx$$

STEP 3: This in turn implies that

$$\left| \overline{\int} f dx - \underline{\int} f dx \right| \leq \left(\int_a^b f_n + \epsilon_n dx \right) - \left(\int_a^b f_n - \epsilon_n dx \right) = 2\epsilon_n (b - a)$$

(If $a \leq b \leq c \leq d$, then $c - b \leq d - a$)

And taking the limit as $n \to \infty$, since $\epsilon_n \to 0$, we get

$$\overline{\int} f dx = \underline{\int} f dx$$

Hence f is Riemann integrable

STEP 4: Finally, using $|f_n - f| \leq \epsilon_n$ again, which implies $-\epsilon_n \leq f_n - f \leq \epsilon_n$ and integrating on both sides, we get

$$-\epsilon_n (b - a) \leq \int_a^b f_n - f dx \leq \epsilon_n (b - a)$$

$$\left| \int_a^b f_n - \int_a^b f dx \right| \leq \epsilon_n (b - a)$$

And taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \int_a^b f_n dx = \int_a^b f(x)dx \quad \square$$

Non-Example: To show that we really need uniform convergence in the above, consider

$$f_n(x) = nx\left(1 - x^2\right)^n$$

Claim: $f_n \to 0$ pointwise on $[0, 1]$

This is true if $x = 0$ and $x = 1$

And if $0 < x < 1$, this follows because exponential functions go faster to 0 than power functions[2] For example, if $x = \frac{1}{2}$, then

$$f_n\left(\frac{1}{2}\right) = \frac{n}{2}\left(\frac{3}{4}\right)^n \to 0\checkmark$$

Now *if* the above result were true, then we would have $\int_0^1 f_n(x)dx \to \int_0^1 0dx = 0$, but using the $u-$sub $u = 1 - x^2$, it follows that

$$\int_0^1 f_n(x)dx = \int_0^1 nx\left(1 - x^2\right)^n dx = \frac{n}{2n+2} \to \frac{1}{2} \neq 0$$

6. Uniform Convergence and Differentiation

Finally, let's discuss differentiability, which is much more delicate!

Example: Consider $f_n : [-1, 1] \to \mathbb{R}$ defined by

$$f_n(x) = \sqrt{x^2 + \frac{1}{n}}$$

[2] To make this rigorous, you would use that $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$ for positive α and p

Then each f_n is differentiable, but f_n converges uniformly to $f(x) = |x|$, which is not differentiable!

So the uniform limit of differentiable functions need not be differentiable. Even worse, it could converge to a **nowhere** differentiable function!

That said, the result *is* true if you assume, moreover, that the sequence of *derivatives* f_n' converges uniformly:

Theorem: (Differentiability)

(1) Suppose f_n is differentiable on $[a, b]$ and $f_n \to f$ uniformly

(2) Moreover, suppose $f_n' \to g$ uniformly for some function g

(3) Then in fact f is differentiable and $f' = g$.

Note: (3) says in this case $f_n' \to f'$ uniformly and that

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

So heuristically, $f_n \to f \Rightarrow f_n' \to f'$ but under extra precaution

Special case of Proof: IF each f_n' is continuous, then the proof is much easier, because by the fundamental theorem of calculus, we then have:

$$f_n(x) = f_n(a) + \int_a^x f_n'(t)dt \to f(a) + \int_a^x g(t)dt$$

And since $f_n(x) \to f(x)$, we get

$$f(x) = f(a) + \int_a^x g(t)dt$$

And by the FTC again, we have $f' = g$ and we would be done.