LECTURE 10: PARTIAL DERIVATIVES

1. Partial Derivatives

Video: Partial Derivatives

Welcome to the Master Sword of Calculus, the one that will help you survive your multivariable adventure: Partial Derivatives! It's *partially* derivatives, but a *whole* lot of fun!

Example 1:

Consider $f(x, y) = x^2y + xy^2$

Goal: Find the derivative of f.

You might ask: How is this even possible, since f depends on two variables? The trick is to differentiate it with respect to each variable separately:

Notation:

$$f_x = \frac{\partial f}{\partial x} = \text{ Derivative of } f \text{ with respect to } x$$
$$f_y = \frac{\partial f}{\partial y} = \text{ Derivative of } f \text{ with respect to } y$$

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All this means is that, for f_x, you differentiate f with respect to x, treating y as a constant (and vice-versa for f_y)

$$f(x,y) = \left(x^2\right) y + (x)\, y^2$$
$$f_x(x,y) = (2x)y + (1)y^2 = 2xy + y^2$$
$$f_y(x,y) = x^2(1) + x(2y) = x^2 + 2xy$$

Example 2:

$f(x,y) = x^4 + 5xy^3$, find f_x, f_y

$$f_x = \left(x^4 + 5xy^3\right)_x = 4x^3 + 5y^3$$
$$f_y = \left(x^4 + 5xy^3\right)_y = 0 + 5x(3y^2) = 15xy^2$$

Small Application: If $V = f(T,P)$ is volume as a function of temperature T and pressure P, then $\frac{\partial f}{\partial T}$ is the change in volume as temperature increases, and $\frac{\partial f}{\partial P}$ is the change in volume as pressure increases.

Of course, can also do it with several variables:

Example 3:

$f(x,y,z) = xe^{yz}$, find f_x, f_y, f_z

$$f_x = \left(xe^{yz}\right)_x = e^{yz}$$
$$f_y = \left(xe^{yz}\right)_y = xe^{yz}z = xze^{yz}$$
$$f_z = \left(xe^{yz}\right)_z = xe^{yz}y = xye^{yz}$$

Note: For an extra fun problem, check out the following video:

Video: Chain Rule Surprise

Just like for ordinary derivatives, we can also evaluate partial derivatives at a point.

Example 4:

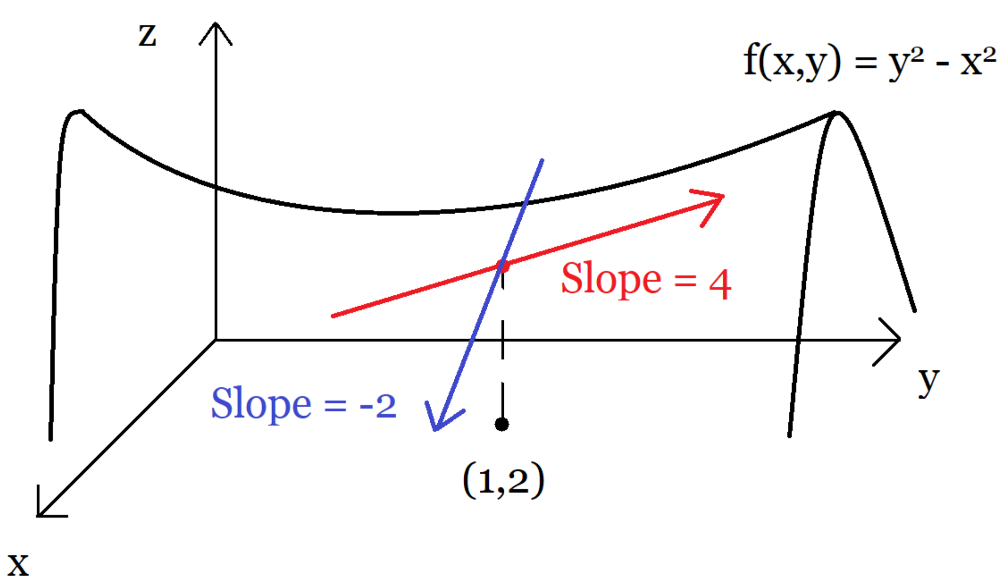
Let $f(x,y) = y^2 - x^2$, find $f_x(1,2)$ and $f_y(1,2)$

$$f_x(x,y) = -2x \Rightarrow f_x(1,2) = -2(1) = -2$$
$$f_y(x,y) = 2y \Rightarrow f_y(1,2) = 2(2) = 4$$

2. Graphical Interpretation

Now that we've seen how to calculate partial derivatives, let's figure out what they really mean!

Back to: $f(x,y) = y^2 - x^2$ (saddle)



Notice in the picture that f is decreasing in the x-direction and increasing in the y-direction, and in fact:

Interpretation:

$f_x(1,2) =$ Slope of tangent line of f at $(1,2)$ in the $x-$direction
$f_y(1,2) =$ Slope of tangent line of f at $(1,2)$ in the $y-$direction

Using this, you can show that:

Actual Definition

$$f_x(1,2) = \lim_{h \to 0} \frac{f(1+h,2) - f(1,2)}{h}$$
$$f_y(1,2) = \lim_{h \to 0} \frac{f(1,2+h) - f(1,2)}{h}$$

3. Higher-Order Partial Derivatives

Just like Pringles, once you pop, you never stop! In this case, once you differentiate once, you want to do it again!

Example 5:

$f(x,y) = x^4 + 2xy^3 + y^4$

(a) Calculate f_x, f_y

$$f_x = 4x^3 + 2y^3$$
$$f_y = 2x(3y^2) + 4y^3 = 6xy^2 + 4y^3$$

(b) Calculate $f_{xx}, f_{xy}, f_{yx}, f_{yy}$

$$
\begin{aligned}
f_{xx} &\overset{DEF}{=} (f_x)_x = (4x^3 + 2y^3)_x = 12x^2 \\
f_{xy} &\overset{DEF}{=} (f_x)_y = (4x^3 + 2y^3)_y = 6y^2 \\
f_{yx} &= (f_y)_x = (6xy^2 + 4y^3)_x = 6y^2 \\
f_{yy} &= (f_y)_y = (6xy^2 + 4y^3)_y = 6x(2y) + 12y^2 = 12xy + 12y^2
\end{aligned}
$$

Notice: $f_{xy} = f_{yx}$ and in fact this is always true!

Clairaut's Theorem

$$f_{xy} = f_{yx}$$

Fun Fact: Clairaut had 19 siblings! He wrote his first paper when he was 12.

Note: Technically need f_{xy} and f_{yx} continuous, there are pathological counterexamples, as in the following video:

Video: $f_{xy} \neq f_{yx}$

Consequences:

$$
\begin{aligned}
f_{xyz} &= f_{zyx} = f_{yzx} = \cdots \\
f_{zzyzx} &= f_{xyzzz} = \cdots
\end{aligned}
$$

4. Implicit Differentiation

Video: Implicit Differentiation

In Calculus 1, you learned how to differentiate implicit functions, like $x^2y + y^3 = 2x$. Here we're able to do the same:

Example 6:

Find $\frac{\partial z}{\partial x}$, where

$$\underbrace{x^3 + y^3 + z^3 = 6xyz}_{\text{Implicit Function}}$$

Differentiate with respect to x:

$$\begin{aligned}\left(x^3 + y^3 + z^3\right)_x &= (6xyz)_x \\ 3x^2 + 3z^2\left(\frac{\partial z}{\partial x}\right) &= 6yz + 6xy\left(\frac{\partial z}{\partial x}\right)\end{aligned}$$

Now just solve this for $\frac{\partial z}{\partial x}$:

$$\begin{aligned}3z^2\left(\frac{\partial z}{\partial x}\right) - 6xy\left(\frac{\partial z}{\partial x}\right) &= 6yz - 3x^2 \\ \left(3z^2 - 6xy\right)\left(\frac{\partial z}{\partial x}\right) &= 6yz - 3x^2 \\ \frac{\partial z}{\partial x} &= \frac{6yz - 3x^2}{3z^2 - 6xy} = \frac{2yz - x^2}{z^2 - 2xy}\end{aligned}$$

Note: You'll see another way to do this in the section on the Chain Rule

5. Talk PDE to me

As an application, partial derivatives arise naturally in (what are called) Partial Differential Equations (PDEs):

Example 7:

Show that $u(x,y) = e^x \cos(y)$ solves Laplace's Equation:

$$u_{xx} + u_{yy} = 0$$

This is an example of a PDE, which is an equation that relates a function u with one or more of its partial derivatives.

$$\begin{aligned} u_{xx} &= (e^x \cos(y))_{xx} = e^x \cos(y) \\ u_{yy} &= (e^x \cos(y))_{yy} = -\left(e^x \sin(y)\right)_y = -e^x \cos(y) \\ \Rightarrow u_{xx} + u_{yy} &= e^x \cos(y) - e^x \cos(y) = 0 \end{aligned}$$

Aside: Not to say that one math subject is better than the other, but clearly PDE is the best math subject of them all! (and I'm not just saying this because it's my specialty)

Let me just say this: If you can solve all PDE, you can solve the universe, which means: (1) PDE are incredibly hard to solve and (2) They are incredibly powerful!

Example: Laplace's equation (above) gives you the temperature of a metal plate after a really long time.

Many Applications: Heat Equation, Wave Equation, Einstein's PDE (black holes), Schrödinger's Equation (quantum mechanics), Black Scholes PDE (finance), the PDE that gave me the PhD (see below)

For a really cool overview of PDEs, as well a glimpse into my PhD thesis, check out the following videos:

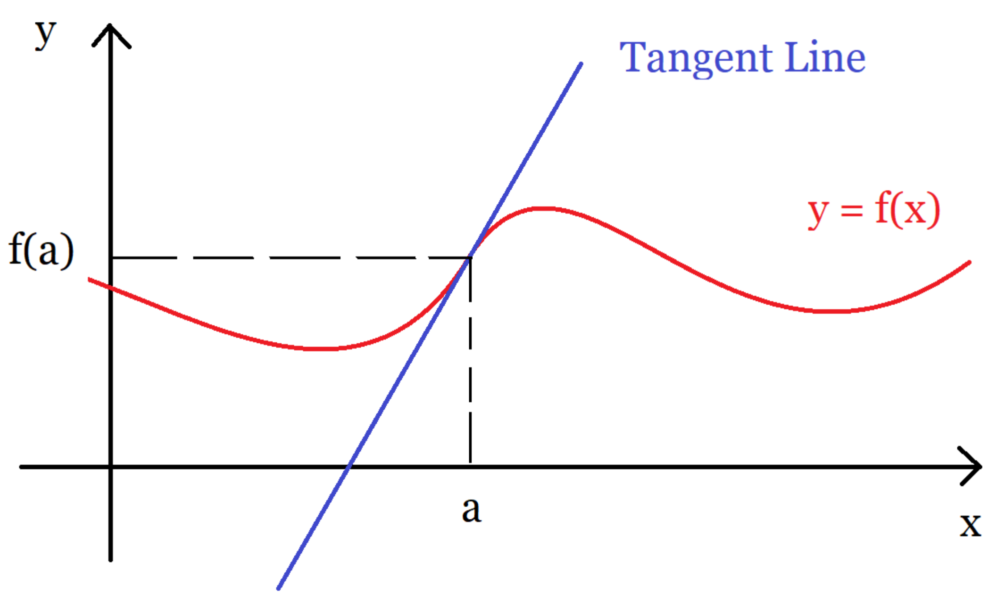
Video: What is a PDE? (optional)

Video: The PDE that gave me the PhD (optional)

6. Tangent Planes

Video: Tangent Planes

Just like derivatives lead to tangent lines, partial derivatives lead to tangent planes.



Recall

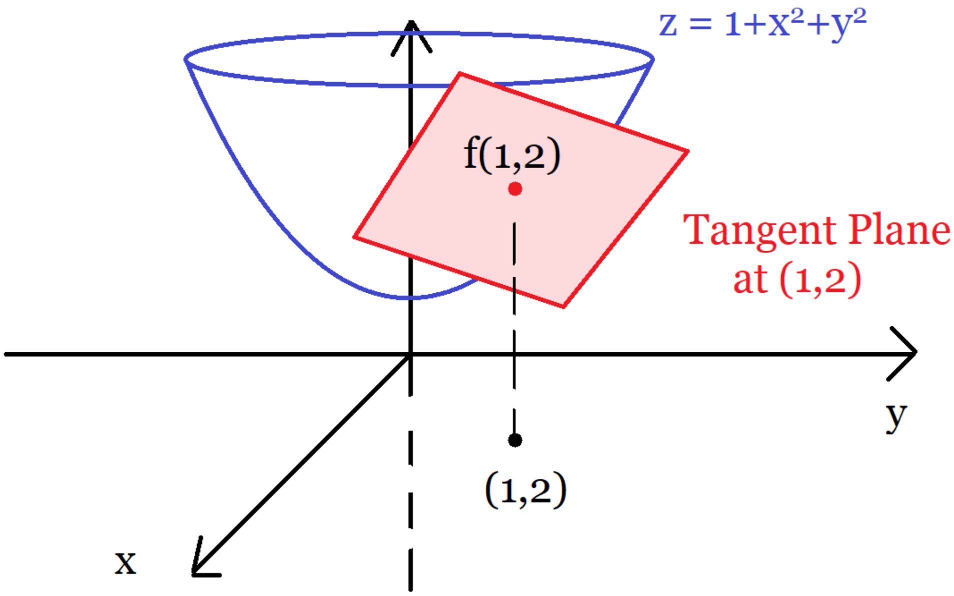
The equation of the tangent line of f at $x = a$ is:

$$y - f(a) = f'(a)(x - a)$$

It's the same idea in 2 dimensions, except tangent lines become *tangent planes*.

Example 8:

Find the equation of the tangent plane of $f(x, y) = 1 + x^2 + y^2$ at $(1, 2)$



Tangent Plane

The equation of the tangent plane of f at $(1, 2)$ is:

$$z - f(1, 2) = f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2)$$

(This is just the point-slope formula, but with two variables. Intuitively this formula makes sense if you think of f_x and f_y as the slopes in the x and y directions).

$$\text{Here: } f(1, 2) = 1 + 1^2 + 2^2 = 6$$
$$f_x(x, y) = 2x \Rightarrow f_x(1, 2) = 2(1) = 2$$
$$f_y(x, y) = 2y \Rightarrow f_y(1, 2) = 2(2) = 4$$

Answer: $z - 6 = 2(x - 1) + 4(y - 2)$