LECTURE 10: FOURIER TRANSFORM

Today: We'll discuss a continuous analog of Fourier series, called the Fourier Transform

1. Definition and Properties

Motivation: If f is a function of period 1, then

$$\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx$$

(The analog of $\frac{1}{2\pi}$ here is $\frac{1}{1} = 1$)

Questions: Is there a continuous analog of this, where n is replaced by a real number? And what if f is not periodic?

Yes there is, and it's called the **Fourier Transform:**

Definition:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx$$

Remarks:

(1) Sometimes this is written as $\mathcal{F}(f)$

(2) $\hat{f}$ is a function of ξ (frequency variable) and not of x (spatial variable)

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(3) The improper integral is defined in the following sense

$$\int_{-\infty}^{\infty} g(x)dx = \lim_{N\to\infty} \int_{-N}^{N} g(x)dx$$

(4) An interesting special case is $\hat{f}(0) = \int_{-\infty}^{\infty} f(x)dx$

(5) f is not necessarily periodic here

Does this work for any function f? No, not even for $f(x) = 1$. We will discuss the appropriate function space below.

Immediate Properties:

(1) (Translation) $\widehat{f(x+h)} = \hat{f}(\xi)e^{2\pi i h\xi}$

(2) (Translation) $\widehat{f(x)e^{-2\pi i x h}} = \hat{f}(\xi + h)$

(3) (Dilation) If $\delta > 0$ then $\widehat{f(\delta x)} = \frac{1}{\delta}\hat{f}\left(\frac{\xi}{\delta}\right)$

This means for example that if $g(x) = f(x+h)$ then $\hat{g}(\xi) = \hat{f}(\xi)e^{2\pi i h\xi}$

For example, if $\delta = 2$ this says $\widehat{f(2x)} = \frac{1}{2}\hat{f}\left(\frac{\xi}{2}\right)$, so the Fourier transform turns compression into stretching, and vice-versa.

(The properties follow from the definition and/or u-subs)

2. The Schwartz Space

In order for the Fourier transform to be well-defined, we need f to go to 0 very fast at $\pm\infty$

Intuitively: We want all the derivatives $f, f^{'}, f^{''}$ etc. not only to be bounded, but also go faster to 0 than any power function $\frac{1}{x}, \frac{1}{x^2}, \frac{1}{x^3}$ etc.

Definition: $f \in \mathcal{S}(\mathbb{R})$ (Schwartz Space) if f is infinitely differentiable and for all k and n

$$\sup_{x \in \mathbb{R}} |x|^k \left| f^{(n)}(x) \right| < \infty$$

(The sup could depend on k and n)

For example, with $n = 0$ this means that $|f(x)| \leq \frac{C_k}{|x|^k}$ for all x and all k

Example: $e^{-x^2} \in \mathcal{S}(\mathbb{R})$, but also functions that are 0 outside a bounded interval

Notice that if $f \in \mathcal{S}$ then $f' \in \mathcal{S}$ and $xf \in \mathcal{S}$, and (see below) $\hat{f} \in \mathcal{S}$

Finally, note that this is just a sufficient condition, there are non-Schwartz functions for which $\hat{f}$ is defined

3. Derivatives and Fourier Transforms

The Fourier transform turns differentiation into multiplication, in the following sense:

Fact: [Differentiation]

(1) $\widehat{f'(x)} = (2\pi i \xi)\hat{f}(\xi)$

(2) $-\widehat{2\pi i x f}(x) = \frac{d}{d\xi}\hat{f}(\xi)$

In particular, it turns differential equations into algebra equations! This is why they're so useful in ODE and PDE.

Proof of (1)**:** Integrating by parts with respect to x gives

$$\int_{-N}^{N} f'(x)e^{-2\pi i x\xi}dx = \left[f(x)e^{-2\pi i x\xi}\right]_{-N}^{N} + 2\pi i\xi \int_{-N}^{N} f(x)e^{-2\pi i x\xi}dx$$

Letting $N \to \infty$ gives the result. The boundary terms are 0 because $\left|e^{-2\pi i x\xi}f(x)\right| = \left|e^{-2\pi i x\xi}\right| |f(x)| = |f(x)| \to 0$ as $x \to \pm\infty$, because f is Schwartz

Proof-Sketch of (2)[1] Follows from writing

$$\frac{\hat{f}(\xi+h)-\hat{f}(\xi)}{h} - \left(\widehat{-2\pi i x f}(\xi)\right) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}\left[\frac{e^{-2\pi i x h}-1}{h} + 2\pi i x\right]$$

And splitting up the integral into two regions, one where $|x|$ is large (where we can use that $f(x)$ and $xf(x)$ are Schwartz) and one where $|x|$ is small, where we can use that

$$\lim_{h\to 0} \frac{e^{-2\pi i x h}-1}{h} + 2\pi i x = -2\pi i x + 2\pi i x = 0$$

Corollary: If $f \in \mathcal{S}$ then $\hat{f} \in \mathcal{S}$

Why? First note that whenever $g \in \mathcal{S}$ then $\hat{g}$ is bounded because

$$|\hat{g}(\xi)| = \left|\int_{-\infty}^{\infty} g(x)e^{2\pi i x\xi}dx\right| \le \int_{-\infty}^{\infty} |g(x)| \left|e^{2\pi i x\xi}\right| dx = \int_{-\infty}^{\infty} |g(x)| \le C$$

But then $\xi\hat{f}(\xi)$ is bounded because it's just the Fourier transform of $\frac{1}{2\pi i}f'(x) \in \mathcal{S}$ and so is $\frac{d}{d\xi}\hat{f}(\xi)$ because it's the Fourier transform of

[1] See Stein and Shakarchi, Prop 1.2(v) in Chapter 5 for details

$-2\pi i x f(x) \in \mathcal{S}$, and you can use this to show that $\xi^k \left| \left(\hat{f}\right)^{(n)} (\xi) \right|$ is always bounded, so $\hat{f} \in \mathcal{S}$ □

4. Self-Adjointness

The Fourier transform has an interesting self-adjointness property:

Fact: If $f, g \in \mathcal{S}$ then

$$\int_{-\infty}^{\infty} \hat{f}(x)g(x)dx = \int_{-\infty}^{\infty} f(y)\hat{g}(y)dy$$

Note: Compare this to $\langle Tx, y \rangle = \langle x, Ty \rangle$ if T is self-adjoint

Proof:

$$\begin{aligned}
\int_{-\infty}^{\infty} \hat{f}(x)g(x)dx &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(y)e^{-2\pi i x y} dy \right) g(x)dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)g(x)e^{-2\pi i x y} dy dx \\
&\overset{\text{FUB}}{=} \int_{-\infty}^{\infty} f(y) \left(\int_{-\infty}^{\infty} g(x)e^{-2\pi i x y} dx \right) dy \\
&= \int_{-\infty}^{\infty} f(y)\hat{g}(y)dy
\end{aligned}$$

(The use of Fubini is justified because f and g are Schwartz)

5. Convolution

We can generalize the notion of convolution to functions on $\mathbb{R}$:

Definition:

$$(f \star g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

Again, f and g are not necessarily periodic. This definition is more widely used in math than the previous one.

Facts:

(1) If $f \in \mathcal{S}$ and $g \in \mathcal{S}$ then $f \star g \in \mathcal{S}$

(2) $f \star g = g \star f$

(3) $\widehat{f \star g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$

The proofs of (2) and (3) are identical to the one in the periodic case (where you use Fubini)

6. The Fourier Inversion Formula

One of the cornerstone theorems in the theory of Fourier transforms is the Fourier inversion formula, which says:

Theorem: [Fourier Inversion]

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi$$

In other words, f is its own Fourier transform! (provided you use $2\pi i$ instead of $-2\pi i$), so in some sense, the Fourier transform is a (sort of) bijection from $\mathcal{S}$ to $\mathcal{S}$

Some preliminary Facts:

(1) If $f(x) = e^{-\pi x^2}$ then $\hat{f}(\xi) = f(\xi)$

(2) If $G_\delta(x) = e^{-\pi\delta x^2}$ then $\widehat{G_\delta}(\xi) = \frac{1}{\sqrt{\delta}} e^{-\frac{\pi x^2}{\delta}} =: K_\delta(\xi)$

The first follows from using the definition and completing the square, and the second one follows from the Dilation property

Proof-Sketch of Fourier Inversion:[2]

STEP 1: First assume $x = 0$ and show

$$f(0) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i(0)\xi} d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi) d\xi$$

Let $G_\delta = e^{-\pi\delta x^2}$ and K_δ as above, then by self-adjointedness, we have

$$\int_{-\infty}^{\infty} f(x) \underbrace{K_\delta(x)}_{\hat{G}_\delta} dx = \int_{-\infty}^{\infty} \hat{f}(\xi) G_\delta(\xi) d\xi$$

STEP 2: **Left term:** By symmetry, $K_\delta(x) = K_\delta(-x)$ and so the integral on the left can be written as

$$\int_{-\infty}^{\infty} f(x) K_\delta(-x) dx = (f \star K_\delta)(0)$$

It can be shown that $\{K_\delta\}$ is a family of Good Kernels and so by a result from the homework, it follows that $(f \star K_\delta) \to f$ uniformly as $\delta \to 0$, and in particular

[2] see Stein and Shakarchi, Theorem 1.9 in Chapter 5

$$(f \star K_\delta)(0) \to f(0)$$

So the left hand side indeed converges to $f(0)$

STEP 3: Right Term

Using $G_\delta = e^{-\pi\delta x^2}$ it follows that as $\delta \to 0$ we have

$$\int_{-\infty}^{\infty} \hat{f}(\xi) G_\delta(\xi) d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{-\pi\delta\xi^2} d\xi \to \int_{-\infty}^{\infty} \hat{f}(\xi) d\xi$$

Combining the two we get

$$f(0) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

STEP 4: In the general case, if x is fixed, let $F(y) = f(y + x)$ then by the $x = 0$ result above and the translation property, we get

$$f(x) = F(0) = \int_{-\infty}^{\infty} \hat{F}(\xi) d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \quad \square$$

7. Plancherel's Formula

As a corollary, we obtain Plancherel's Formula, which says that the Fourier transform is an isometry:

Theorem: [Plancherel]

$$\text{If } f \in \mathcal{S} \text{ then } \left\| \hat{f} \right\| = \| f \|$$

Proof: Define $f^s(x) = \overline{f(-x)}$ then can show that $\widehat{f^s}(\xi) = \overline{\hat{f}(\xi)}$.

Let $h = f \star f^s$ then by the Fourier inversion formula with $x = 0$ we get

$$h(0) = \int_{-\infty}^{\infty} \hat{h}(\xi) d\xi$$

$$\begin{aligned} h(0) = \left(f \star f^s\right)(0) = \int_{-\infty}^{\infty} f(x) f^{(s)}(-x) dx &= \int_{-\infty}^{\infty} f(x) \overline{f(-(-x))} dx \\ &= \int_{-\infty}^{\infty} |f(x)|^2 \, dx \end{aligned}$$

$$\hat{h}(\xi) = \hat{f}(\xi) \hat{f}^s(\xi) = \hat{f}(\xi) \overline{\hat{f}(\xi)} = \left| \hat{f}(\xi) \right|^2$$

Combining the two, we get

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} \left| \hat{f}(\xi) \right|^2 d\xi \quad \square$$