LECTURE 11: SUBSEQUENCES (II)

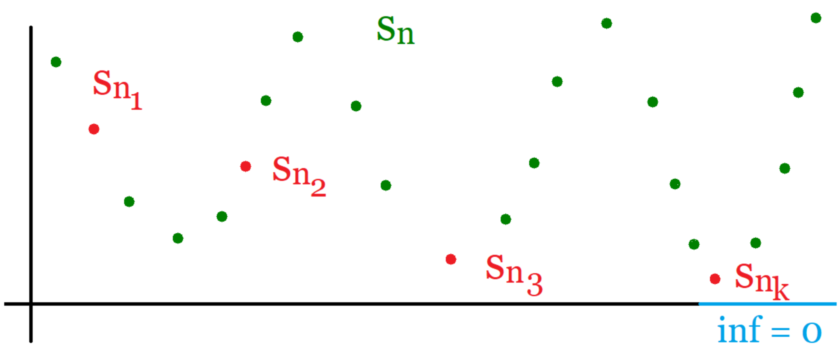
1. Another Inductive Construction

Video: Inductive Construction 2

Let's do another inductive construction, but with subsequences:

Example 4:

Suppose (s_n) is a positive sequence with $\inf \{s_n \mid n \in \mathbb{N}\} = 0$. Show that (s_n) has a decreasing subsequence (s_{n_k}) converg. to 0



Goal: Construct a subsequence (s_{n_k}) with $s_{n_{k+1}} < s_{n_k}$ for all k and

$$0 < s_{n_k} < \frac{1}{k}$$

STEP 1: Want $s_{n_1} < 1$, but notice that $1 > 0 = \inf \{s_n \mid n \in \mathbb{N}\}$

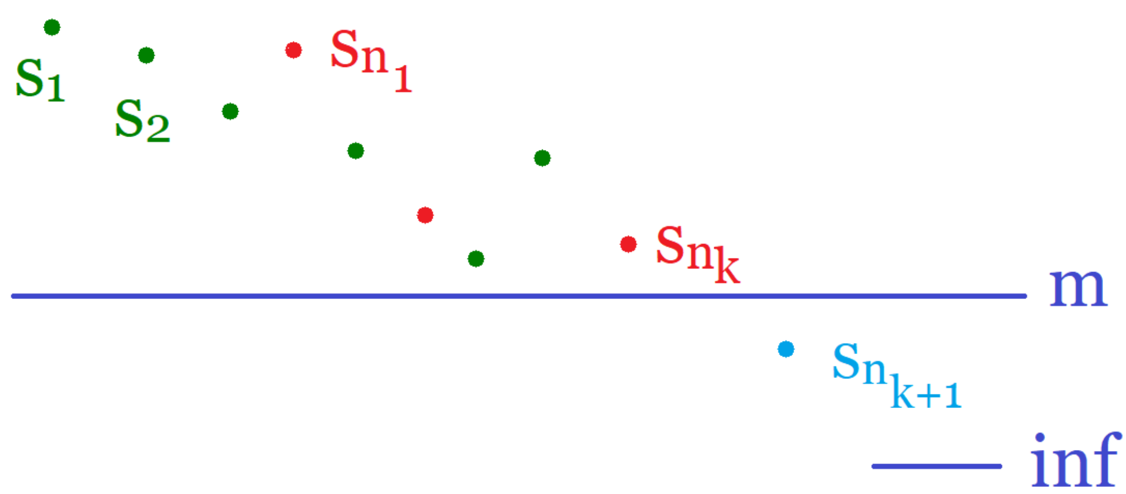
Date: Tuesday, October 5, 2021.

So by definition of inf, there is some s_{n_1} with $s_{n_1} < 1$. ✓

STEP 2: Suppose we found $s_{n_1} > s_{n_2} > \cdots > s_{n_k}$ such that $s_{n_j} < \frac{1}{j}$ for all $j = 1, 2, \ldots, k$, and want to find $s_{n_{k+1}}$ with $s_{n_{k+1}} < s_{n_k}$ and $s_{n_{k+1}} < \frac{1}{k+1}$

Note: You can't directly apply the argument above because it doesn't guarantee that $s_{n_{k+1}} < s_{n_k}$ and you don't even know whether $n_{k+1} > n_k$. To get around this, consider

$$m = \min\left\{\frac{1}{k+1}, s_1, s_2, \ldots, s_{n_k}\right\} > 0 \text{ (not a typo)}$$



Then $m > 0 = \inf\{s_n \mid n \in \mathbb{N}\}$, so by definition of inf, there is $s_{n_{k+1}}$ such that

$$s_{n_{k+1}} < m = \min\left\{\frac{1}{k+1}, s_1, s_2, \ldots, s_{n_k}\right\}$$

Therefore we get $s_{n_{k+1}} < \frac{1}{k+1}$, $s_{n_{k+1}} < s_{n_k}$ and finally $n_{k+1} > n_k$ because $s_{n_{k+1}}$ is smaller than (hence different from) all the terms preceding it, so it cannot be equal to any of its previous terms ✓

STEP 3: Hence, by the inductive construction, we have found a subsequence (s_{n_k}) such that $s_{n_{k+1}} < s_{n_k}$ (decreasing) and $0 < s_{n_k} < \frac{1}{k}$ for all k. So, by the squeeze theorem, we get $\lim_{k\to\infty} s_{n_k} = 0$ □

2. Monotone Subsequence

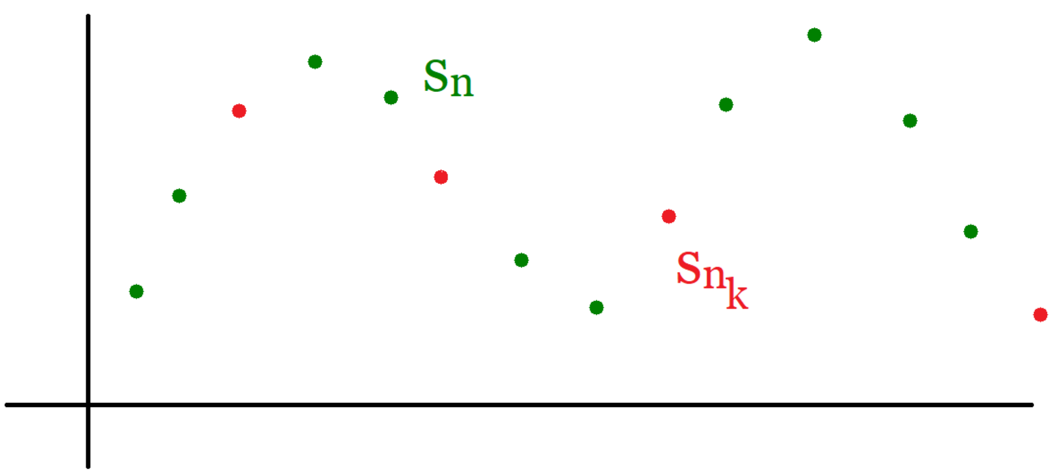
Video: Monotone Subsequence

Here's a miraculous fact about subsequences, with an elegant proof:

Theorem:

Every sequence (s_n) has a monotonic subsequence

(Monotonic means either nondecreasing or nonincreasing)

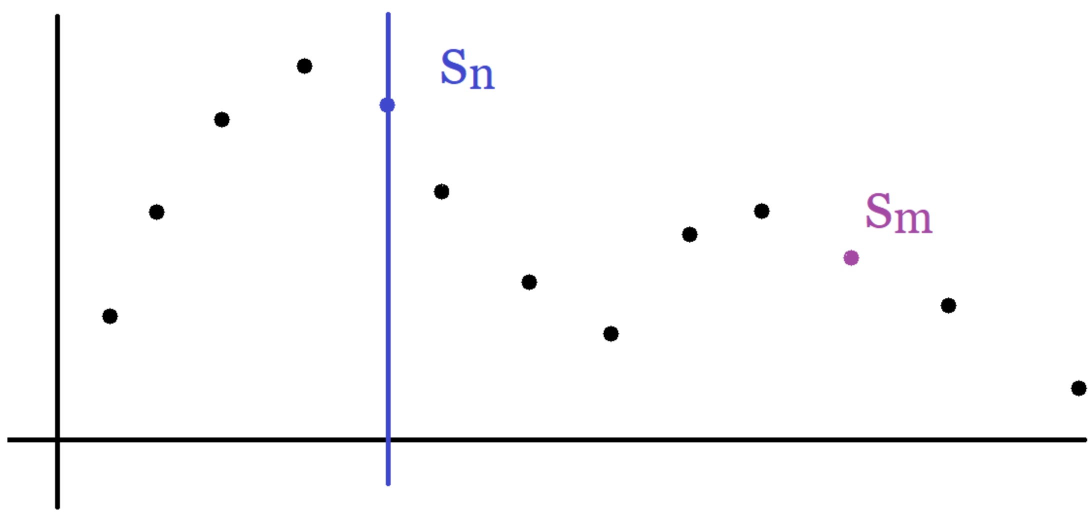


Note: There absolutely no assumptions about (s_n). It could be divergent, it could be unbounded, it could be wild! That's what makes this theorem so powerful!

Proof: Neat!

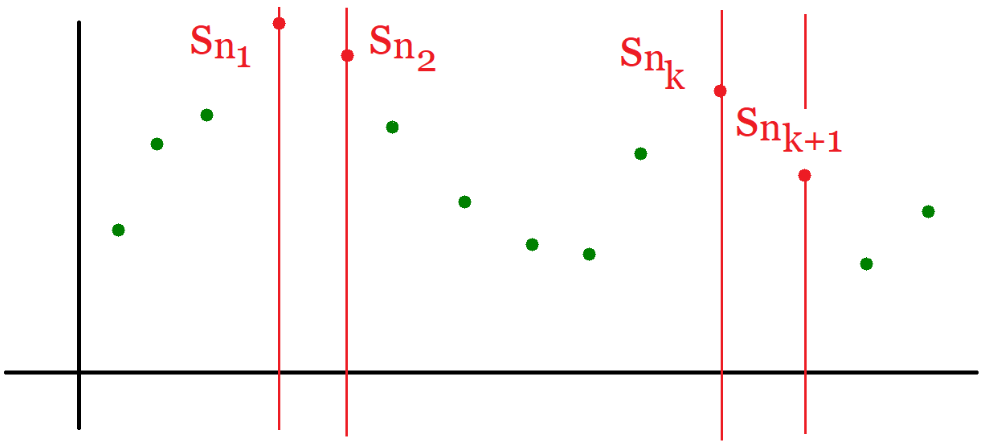
Definition:

We say the number s_n is **dominant** if for all $m > n$, $s_m < s_n$



Kind of like decreasing, except here we're fixing n. Think "Everything is going downhill after s_n" (kind of like the stock market crashing right after you buy some stocks)

Case 1: Suppose there are infinitely many dominant terms, let's denote them (in order) $s_{n_1}, s_{n_2}, \ldots$

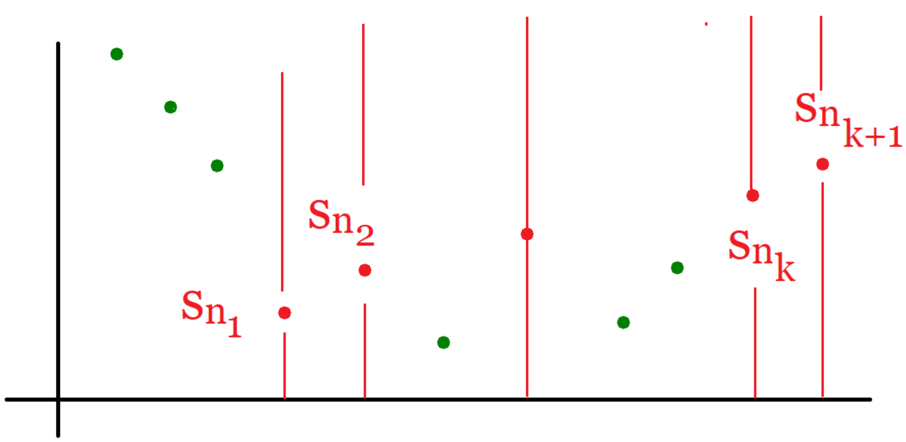


Claim: $(s_{n_k}) = (s_{n_1}, s_{n_2}, \dots)$ is decreasing

But if $n_{k+1} > n_k$, then since s_{n_k} is dominant, we have $s_{n_{k+1}} < s_{n_k}$ ✓

Case 2: There are only finitely many dominant terms.

This is lit! We'll fail so hard at constructing a decreasing sequence that we're actually constructing an increasing one ☺



Let n_1 be larger than the largest dominant term. Since n_1 is not dominant, by definition there must be $n_2 > n_1$ such that $s_{n_2} \geq s_{n_1}$.

Since n_2 is not dominant (we're already passed the dominant terms), there must be $n_3 > n_2$ such that $s_{n_3} \geq s_{n_2}$.

Inductively, since n_k is not dominant, there must be $n_{k+1} > n_k$ such that $s_{n_{k+1}} \geq s_{n_k}$.

Therefore we have inductively constructed a subsequence (s_{n_k}) with $s_{n_{k+1}} \geq s_{n_k}$ for all k, so this subsequence is nondecreasing ✓ □

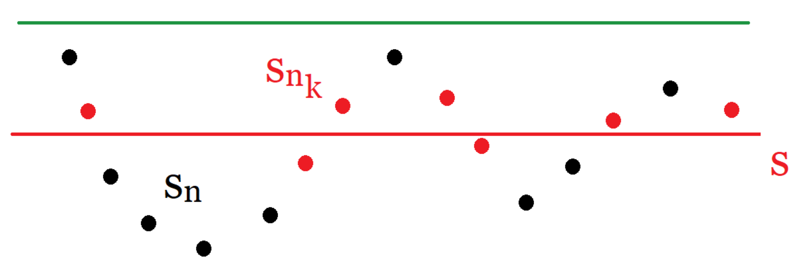
3. The Bolzano-Weierstrass Theorem

Video: Bolzano-Weierstraß Theorem

We are now finally ready for the cornerstone of Analysis: The celebrated Bolzano-Weierstraß Theorem. It says that, even though sequences may not always converge, we have:

Bolzano-Weierstraß Theorem:

Every bounded sequence (s_n) has a convergent subsequence (s_{n_k})



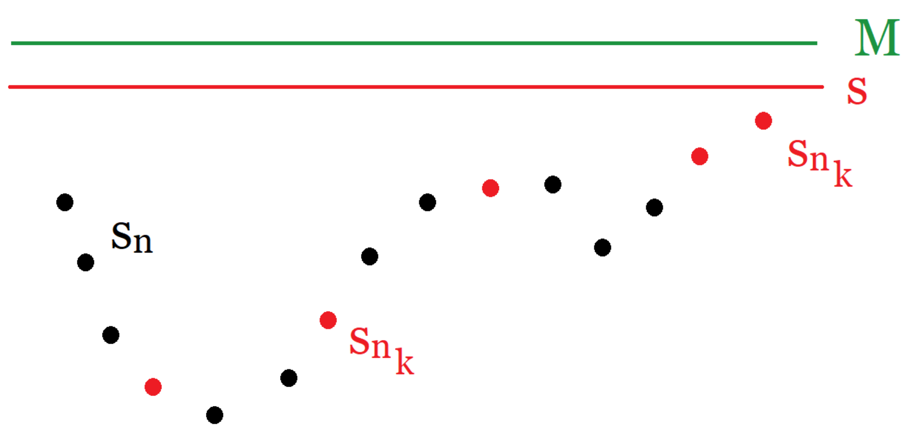
So even if (s_n) doesn't converge, some part of it must converge.

Analogy: Suppose you are trapped in your room and are looking for a pair of scissors. Even though you might never find the pair of scissors, in your search you keep finding car keys you're not looking for. In other words, even though you may not converge to the scissors, part of you (a subsequence) converges to the car keys.

Note: This is wrong if (s_n) is unbounded. For example, no subsequence of $s_n = n$ converges.

Proof: Easy!

Let (s_n) be a bounded sequence. Then, by the above, (s_n) has a monotonic subsequence (s_{n_k}).



Case 1: (s_{n_k}) is nondecreasing

Then (s_{n_k}) is nondecreasing and bounded above (since (s_n) is bounded), so by the Monotone Sequence Theorem (s_{n_k}) converges (see picture above) ✓

Case 2: (s_{n_k}) is nonincreasing

Then (s_{n_k}) is nonincreasing and bounded below, so it converges ✓ □

4. Limsup and Subsequences

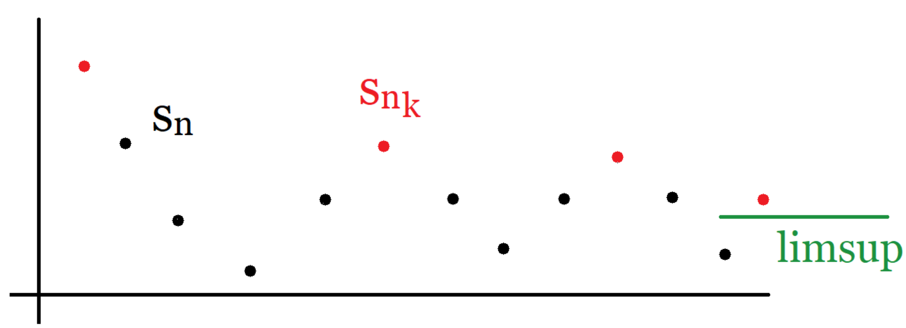
Video: Limsup and Subsequences

More generally, we can say the following:

Theorem:

For any sequence (s_n) there is a subsequence (s_{n_k}) that converges to $\limsup_{n\to\infty} s_n$

In other words, there is always an express train leading to lim sup. This makes lim sup somewhat more concrete; it isn't this weird and abstract concept any more, we actually reach it using subsequences.



Note: The proof is **HARD** and will be omitted, you can check it out in the video if you'd like.

5. Limit Points

Video: Limit Points

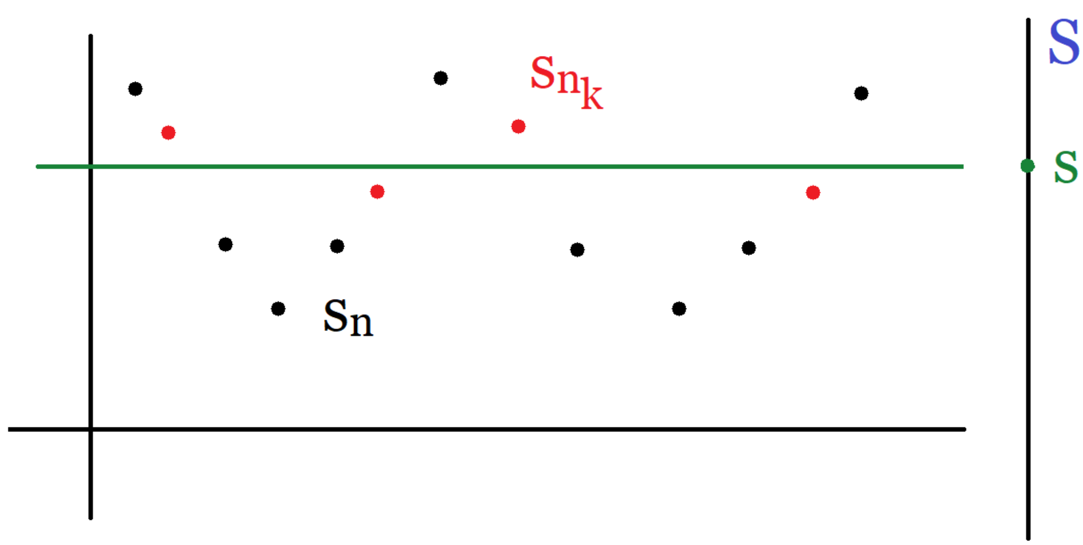
Let's move on with a more fun and useful topic: Limit Points (also known as Subsequential Limits):

Definition:

Let (s_n) be a sequence. Then s is called a **limit point** of (s_n) if there is a subsequence (s_{n_k}) which converges to s.

So s is a limit point if there is an express train going to s.

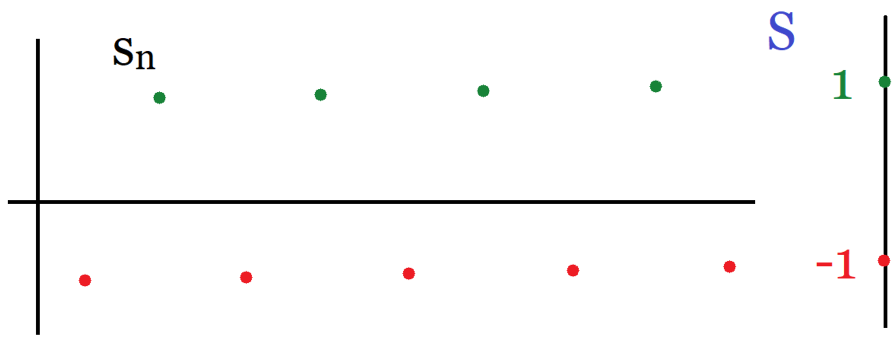
Note: We allow s to be $\pm\infty$. Also, the set of limit points will be denoted by S.



Let's check out some examples of limit points:

Example 1:

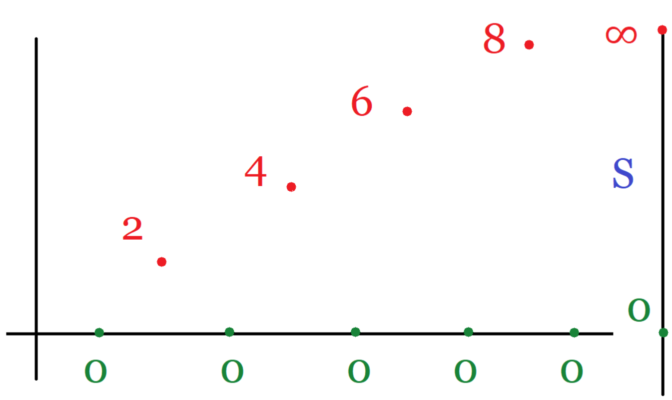
$$s_n = (-1)^n$$



Here the limit points are 1 (a subsequence is $(1, 1, 1, 1, 1, \dots)$) and -1 (a subsequence is $(-1, -1, -1, -1, \dots)$). Hence $\boxed{S = \{-1, 1\}}$

Example 2:

$$s_n = \begin{cases} 0 & \text{if } n \text{ odd} \\ n & \text{if } n \text{ even} \end{cases}$$

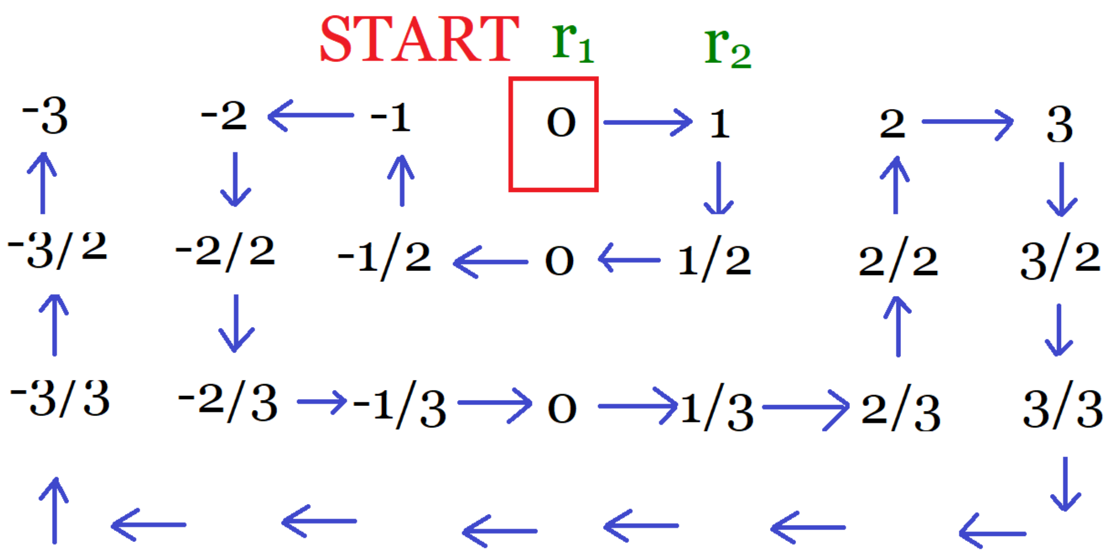


Here 0 is in S (Subsequence: $(0, 0, 0, \dots)$) but also ∞ is in S (Subsequence: $(2, 4, 6, 8, \dots)$), so $\boxed{S = \{0, \infty\}}$.

S can be even crazier than that!

Example 3:

Let (r_n) be the following enumeration of rational numbers



Then we can get arbitrarily close to **every** real number. Moreover, ∞ is a limit point (using the subsequence $(1, 2, 3, \dots)$) and so is $-\infty$ (using the subsequence $(-1, -2, -3, \dots)$), therefore

$$S = \mathbb{R} \cup \{\infty, -\infty\} = \mathbb{R}^{\star}$$

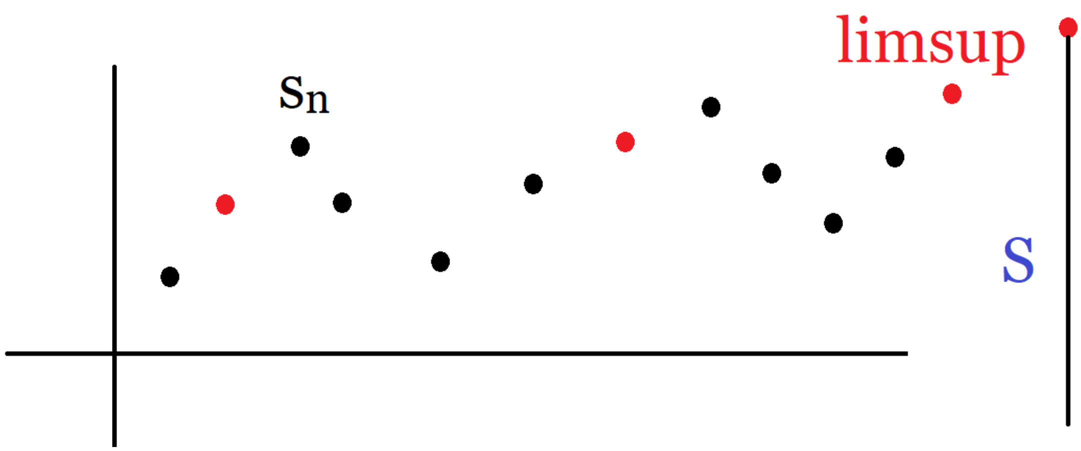
Where $\mathbb{R}^{\star}$ is the *extended* real numbers (that is $\mathbb{R}$ with $\pm\infty$)

6. Limit Point Facts

Let (s_n) be a sequence and S be the set of its limit points.

Fact 1: $S \neq \emptyset$

Why? From the (omitted) proof above, we know that there is a subsequence (s_{n_k}) of (s_n) that converges to $\limsup_{n\to\infty} s_n$.



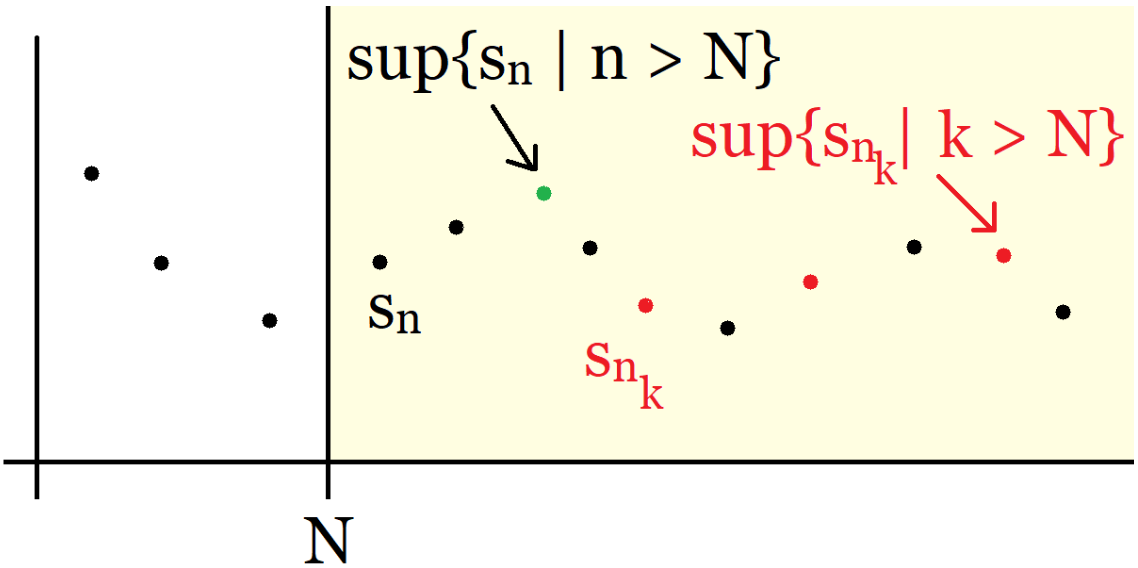
Therefore $\limsup_{n\to\infty} s_n \in S$ and so $S \neq \emptyset$ (it literally contains the number $\limsup s_n$)

Fact 2: $\sup(S) = \limsup_{n\to\infty} s_n$

So limsup is the biggest possible limit point of S. Similar for $\liminf$

Why? Let $s \in S$ be arbitrary. By definition of S, there is a subsequence (s_{n_k}) such that $\lim_{k\to\infty} s_{n_k} = s$.

But now, for each N, consider the sets $\{s_n \mid n > N\}$ and $\{s_{n_k} \mid k > N\}$. The first set has more elements than the second one, since there are more regular stops than express stops.



Therefore, we must have:

$$\sup\{s_n \mid n > N\} \geq \sup\{s_{n_k} \mid k > N\}$$

Hence, taking the limit as $N \to \infty$, we get:

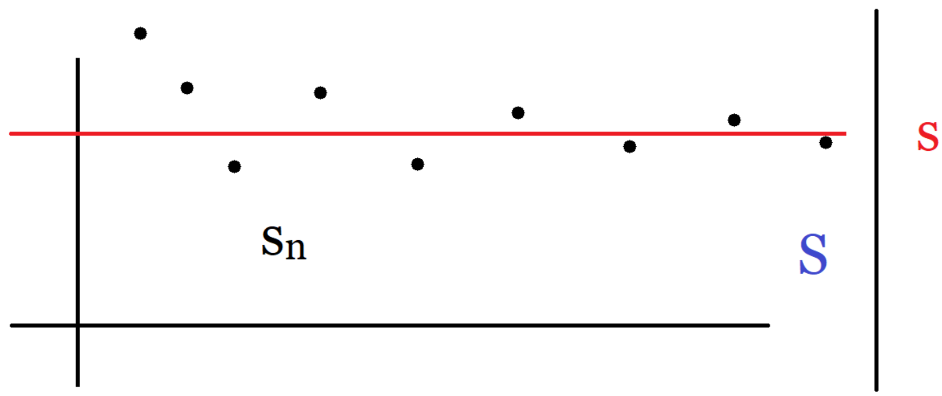
$$\begin{aligned}\limsup_{n\to\infty} s_n &\overset{DEF}{=} \lim_{N\to\infty} \sup\{s_n \mid n > N\} \\ &\geq \lim_{N\to\infty} \sup\{s_{n_k} \mid k > N\} \\ &= \limsup_{k\to\infty} s_{n_k} \\ &= s \text{ (Since } s_{n_k} \text{ converges to } s)\end{aligned}$$

Therefore for all $s \in S$, $s \leq \limsup_{n\to\infty} s_n$, and since s was arbitrary, we get $\sup(S) \leq \limsup_{n\to\infty} s_n$.

On the other hand, by Fact 1 above, we know $\limsup_{n\to\infty} s_n \in S$, and therefore, since for all $s \in S$, $\sup(S) \geq s$, we get $\sup(S) \geq \limsup_{n\to\infty} s_n$.

Therefore $\sup(S) = \limsup_{n\to\infty} s_n$ ✓

Fact 3: $\lim_{n\to\infty} s_n = s \Leftrightarrow S = \{s\}$



Why? ($\Rightarrow$) Suppose $s_n \to s$, then by Fact 2 above, we have:

$$\sup(S) = \limsup_{n\to\infty} s_n = s = \liminf_{n\to\infty} s_n = \inf(S)$$

Hence $\sup(S) = \inf(S) = s$ (the biggest and smallest value of S is s) and so $S = \{s\}$ ✓

($\Leftarrow$) Suppose $S = \{s\}$, but then

$$\limsup_{n\to\infty} s_n = \sup(S) = s = \inf(S) = \liminf_{n\to\infty} s_n$$

Therefore $\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = s$, so by the lim sup squeeze theorem, we get that (s_n) converges to s ✓ □