LECTURE 11: MIDTERM 1 – REVIEW

Welcome to the Midterm 1 review session, and let's start with some fun with planes!

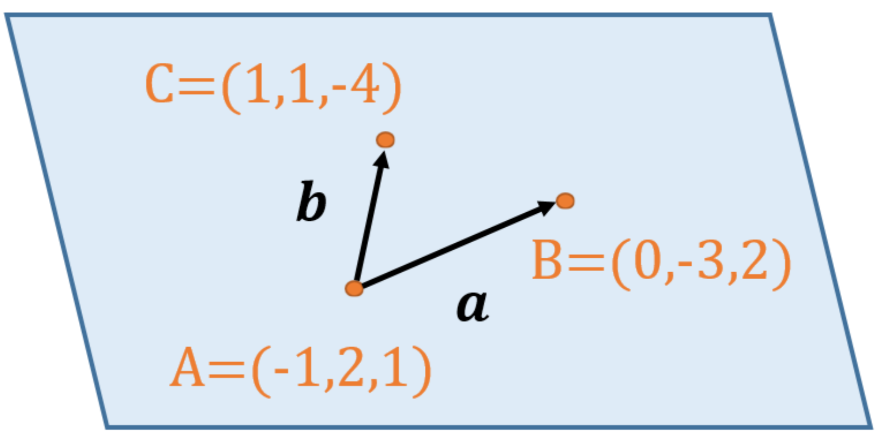
1. Plane Boring?

Example 1:

Find the equation of the plane containing the folowing points:

$$A = (-1, 2, 1), B = (0, -3, 2), C = (1, 1, -4)$$

STEP 1: Picture



STEP 2: Vectors

Date: Wednesday, September 22, 2021.

$$\mathbf{a} = \overrightarrow{AB} = \langle 0-(-1), -3-2, 2-1\rangle = \langle 1, -5, 1\rangle$$
$$\mathbf{b} = \overrightarrow{AC} = \langle 1-(-1), 1-2, -4-1\rangle = \langle 2, -1, -5\rangle$$

STEP 3: Normal Vector

$$\begin{aligned}
\mathbf{n} &= \mathbf{a} \times \mathbf{b} \\
&= \begin{vmatrix} i & j & k \\ 1 & -5 & 1 \\ 2 & -1 & -5 \end{vmatrix} \\
&= \begin{vmatrix} -5 & 1 \\ -1 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 2 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -5 \\ 2 & -1 \end{vmatrix} \mathbf{k} \\
&= \langle 25+1, -(-5-2), -1+10\rangle \\
&= \langle 26, 7, 9\rangle
\end{aligned}$$

STEP 4: Point: $A = (-1, 2, 1)$

STEP 5: Equation

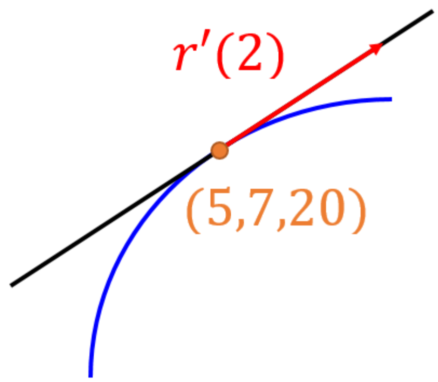
$$26(x+1) + 7(y-2) + 9(z-1) = 0$$

2. Tangent to the Course

Example 2:

Find the parametric equation of the tangent line to the following curve at the point $(5, 7, 20)$

$$r(t) = \left\langle t^2+1, t^3-1, t^4+t^2\right\rangle$$



STEP 1: Find t

$$\left\langle t^2+1, t^3-1, t^4+t^2\right\rangle = \langle 5, 7, 20\rangle$$

Look at the first equation (easiest)

$$t^2+1=5 \Rightarrow t^2=4 \Rightarrow t=\pm 2$$

Check: $t=2$

$$\begin{aligned} t^3-1 &= 2^3-1=7\checkmark \\ t^4+t^2 &= 2^4+2^2=20\checkmark \end{aligned}$$

Hence $t=2$ works

Check: $t=-2$

But then $t^3-1=(-2)^3-1=-9\neq 7$, so $t=-2$ doesn't work.

Therefore we only have $\boxed{\text{t} = 2}$.

STEP 2: Direction Vector:

$$\begin{aligned} r'(t) &= \left\langle 2t, 3t^2, 4t^3+2t\right\rangle \\ r'(2) &= \left\langle 2(2), 3(2)^2, 4(2)^3+2(2)\right\rangle = \langle 4, 12, 36\rangle \end{aligned}$$

STEP 3: Point: $(5, 7, 20)$

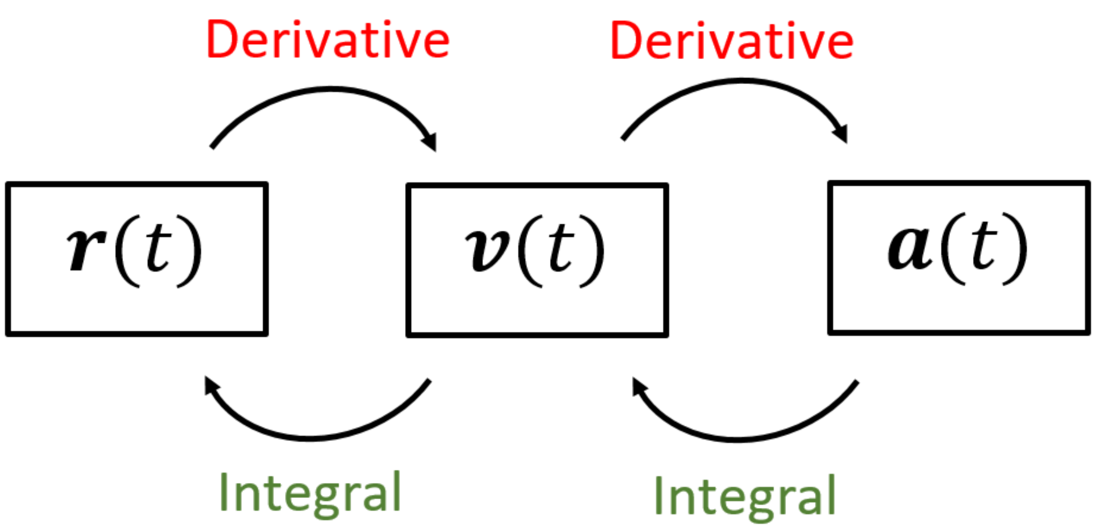
STEP 4: Equation:

$$\begin{cases} x(t) = & 5 + 4t \\ y(t) = & 7 + 12t \\ z(t) = & 20 + 36t \end{cases}$$

3. Acceleration and all that

Example 3:

Find the position $\mathbf{r}(t)$ of a particle, given that its acceleration is $\langle 2, 3, 2 \rangle$ and its velocity at $t = 1$ is $\langle 4, 0, -2 \rangle$ and its initial position is $\langle 1, 2, 3 \rangle$



STEP 1: Find $v(t)$

$$\begin{aligned} a(t) &= \langle 2, 3, 2 \rangle \\ \Rightarrow v(t) &= \int a(t)dt \\ &= \left\langle \int 2, \int 3, \int 2 \right\rangle \\ &= \langle 2t + A, 3t + B, 2t + C \rangle \end{aligned}$$

$v(1) = \langle 2(1) + A, 3(1) + B, 2(1) + C \rangle = \langle 2 + A, 3 + B, 2 + C \rangle = \langle 4, 0, -2 \rangle$

Therefore

$$\begin{cases} 2 + A = 4 \Rightarrow A = 2 \\ 3 + B = 0 \Rightarrow B = -3 \\ 2 + C = -2 \Rightarrow C = -4 \end{cases}$$

Hence $v(t) = \langle 2t + 2, 3t - 3, 2t - 4 \rangle$

STEP 2: Find $\mathbf{r}(t)$

$$\begin{aligned} \mathbf{r}(t) &= \int v(t)dt \\ &= \left\langle \int 2t + 2, \int 3t - 3, \int 2t - 4 \right\rangle \\ &= \left\langle t^2 + 2t + D, \frac{3}{2}t^2 - 3t + E, t^2 - 4t + F \right\rangle \end{aligned}$$

$$\mathbf{r}(0) = \langle D, E, F \rangle = \langle 1, 2, 3 \rangle$$

Hence $D = 1, E = 2, F = 3$ and therefore

$$\mathbf{r}(t) = \left\langle t^2 + 2t + 1, \frac{3}{2}t^2 - 3t + 2, t^2 - 4t + 3 \right\rangle$$

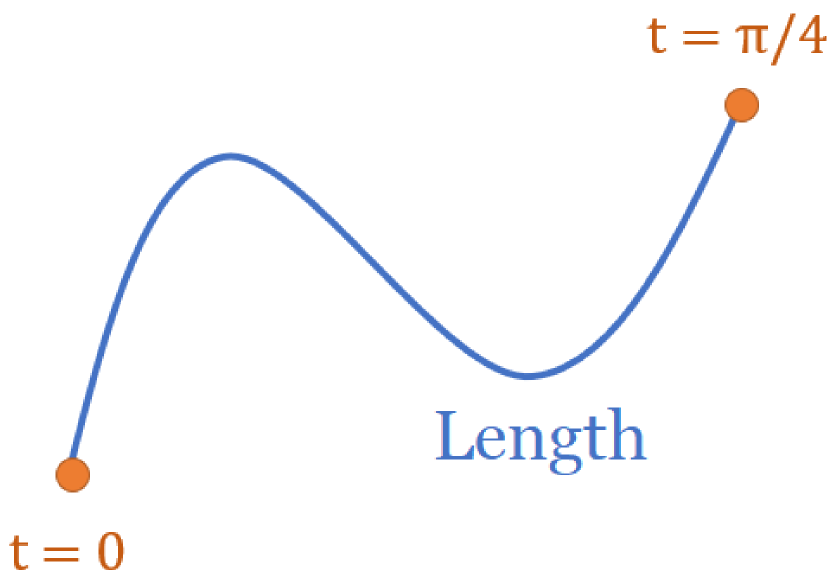
4. This lecture is so long

Let's do a fun problem with arclength!

Example 4:

Find the length of the following curve from $t = 0$ to $t = \frac{\pi}{4}$

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), \ln(\cos(t)) \rangle$$



$$\begin{aligned} \text{Length} &= \int_0^{\frac{\pi}{4}} \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt \\ &= \int_0^{\frac{\pi}{4}} \sqrt{(-\sin(t))^2 + (\cos(t))^2 + \left(\frac{-\sin(t)}{\cos(t)}^2\right)} dt \\ &= \int_0^{\frac{\pi}{4}} \sqrt{\sin^2(t) + \cos^2(t) + \frac{\sin^2(t)}{\cos^2(t)}} dt \end{aligned}$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{1+\tan^2(t)}dt$$
$$= \int_0^{\frac{\pi}{4}} \sqrt{\sec^2(t)}dt$$
$$= \int_0^{\frac{\pi}{4}} \sec(t)dt$$
$$= \left[\ln|\tan(t)+\sec(t)|\right]_0^{\frac{\pi}{4}}$$
$$= \ln\left|\sec\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{4}\right)\right| - \ln|\tan(0)+\sec(0)|$$
$$= \ln\left|\sqrt{2}+1\right| - \ln|0+1|$$
$$= \ln\left(\sqrt{2}+1\right)$$

5. Scratching the Surface

Example 5:

Name the following surfaces

(a) $\frac{x^2}{4}+\frac{y^2}{16}-\frac{z^2}{9}=1$

(b) $z = 2x^2 - 3y^2$

(c) $x = 4y^2 + 3z^2$

(d) $x^2 + y^2 - z^2 = -2$

(e) $z^2 - x^2 - y^2 = 0$

(a) Looks like $x^2 + y^2 - z^2 = 1$. Since it has one minus, it's a hyperboloid of one sheet/dress

(b) Since it's of the form $z =$ something, and there is a minus, it's a hyperbolic paraboloid/saddle

(c) It's similar to $z = x^2 + y^2$ but with x, so it's an (elliptic) paraboloid

(d) Notice this is equivalent to $-x^2 - y^2 + z^2 = 2$. Since there are two minuses (and the 2 is positive), it's a hyperboloid of two sheets/two cups

(e) It's the same as $z^2 = x^2 + y^2$, so it's a cone

Example 6:

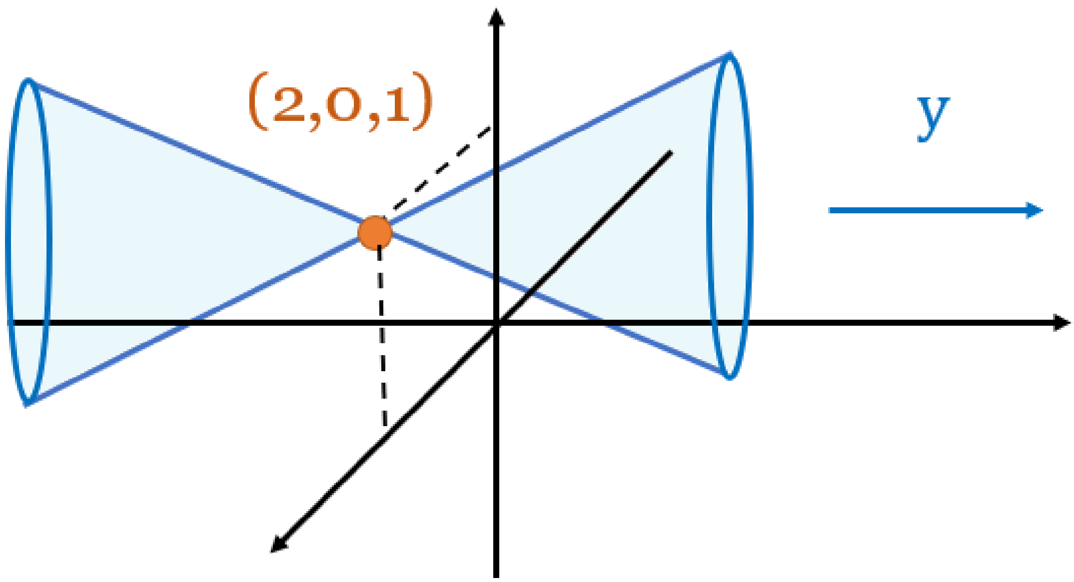
Find the name, center, and direction of the following surface and sketch a rough picture

$$x^2 - y^2 + z^2 = 4x + 2z - 5$$

The trick is to complete the square:

$$\begin{aligned} x^2 - 4x - y^2 + z^2 - 2z + 5 &= 0 \\ (x-2)^2 - 4 - y^2 + (z-1)^2 - 1 + 5 &= 0 \\ (x-2)^2 - y^2 + (z-1)^2 &= 0 \\ y^2 &= (x-2)^2 + (z-1)^2 \end{aligned}$$

This looks similar to $z^2 = x^2 + y^2$, and in fact it's a **cone** in the $y-$direction, centered at $(2, 0, 1)$



6. Parallel, Skew, or Intersecting

Example 7:

Are the following lines parallel, intersecting, or skew?

$$L_1 : \begin{cases} x(t) = t+1 \\ y(t) = t+2 \\ z(t) = t+3 \end{cases} \qquad L_2 : \begin{cases} x(s) = 2s \\ y(s) = s+1 \\ z(s) = 2s+1 \end{cases}$$

The direction vector of L_1 is $\langle 1, 1, 1 \rangle$ and the direction vector of L_2 is $\langle 2, 1, 2 \rangle$ which are not parallel, so the lines either intersect or are skew.

To figure this out, we need to solve all 3 equations.

$$\begin{cases} t+1 = 2s \\ t+2 = s+1 \\ t+3 = 2s+1 \end{cases}$$

The first equation tells us $t = 2s - 1$

Then second equation becomes:

$$\begin{aligned} t+2 &= s+1 \\ 2s-1+2 &= s+1 \\ 2s+1 &= s+1 \\ s &= 0 \end{aligned}$$

And therefore $t = 2s - 1 = 2(0) - 1 = -1$

But then the third equation becomes:

$$t + 3 = 2s + 1 \Rightarrow (-1) + 3 = 2(0) + 1 \Rightarrow 2 = 1$$

Which is a contradiction, and therefore the equation has no solution, and hence the lines are **skew**.