LECTURE 12: LIMSUP PROPERTIES

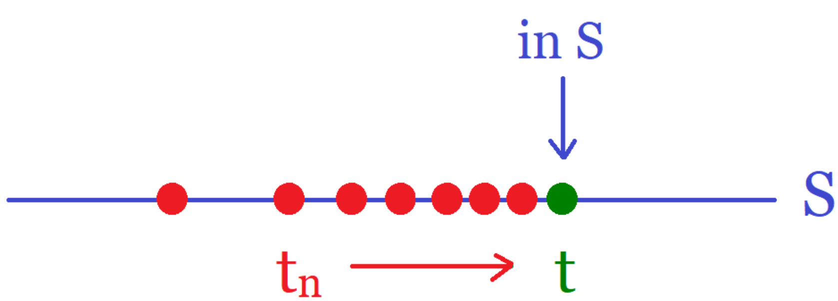
1. Limit Points are Closed

Video: Limit Points are Closed

Let S be the set of limit points of (s_n). Then S isn't just a random set, it has a special structure:

Fact:

If (t_n) is a sequence in S that converges to t, then $t \in S$



Note: In other words, S is a *closed* set

Proof: (assume that (s_n) is bounded, but the fact is true in general)

STEP 1: Suppose (t_k) is a sequence in S that converges to t, want to show that $t \in S$.

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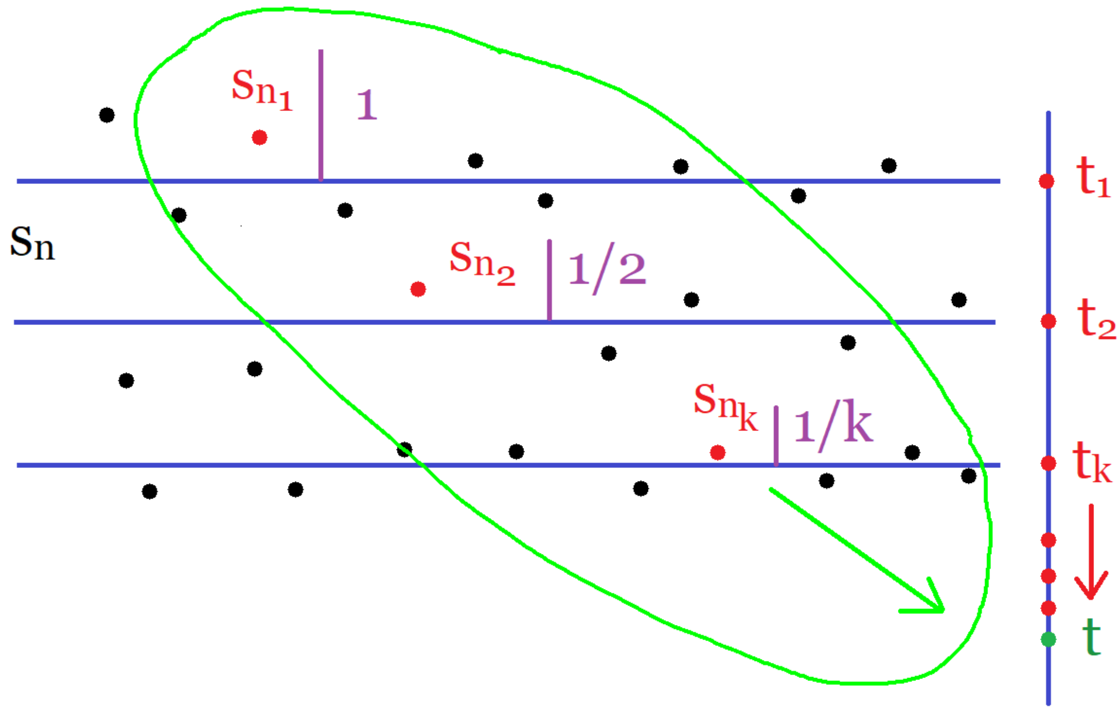


Since $t_1 \in S$, there is a subsequence of (s_n) that converges to t_1. So by the definition of a limit (with $\epsilon = 1$), there is some s_{n_1} with $|s_{n_1} - t_1| < 1$.

Since $t_2 \in S$, by the definition of a limit (with $\epsilon = \frac{1}{2}$), there is some s_{n_2} with $|s_{n_2} - t_2| < \frac{1}{2}$

Note: Can assume $n_2 > n_1$. This is possible since there are infinitely many s_{n_2} as above, so choose one that comes after s_{n_1}

And in general, since $t_k \in S$, there is some s_{n_k} with $|s_{n_k} - t_k| < \frac{1}{k}$, and, as above, choose s_{n_k} with $n_1 < n_2 < \cdots < n_k$

Therefore, we obtain a subsequence $(s_{n_k}) = (s_{n_1}, s_{n_2}, \dots)$.

STEP 2:

Claim: $s_{n_k} \to t$

This is because

$$|s_{n_k} - t| = |s_{n_k} - t_k + t_k - t| \leq |s_{n_k} - t_k| + |t_k - t| < \frac{1}{k} + |t_k - t|$$

But since $t_k \to t$ (by assumption), we get $\frac{1}{k} + |t_k - t| \to 0$ as $k \to \infty$, so by the squeeze theorem, $\lim_{k\to\infty} s_{n_k} = t$. ✓

Thus (s_{n_k}) is a subsequence of (s_n) that converges to t, so $t \in S$ ✓ □

2. Limsup Product Rule

Video: Limsup Product Rule

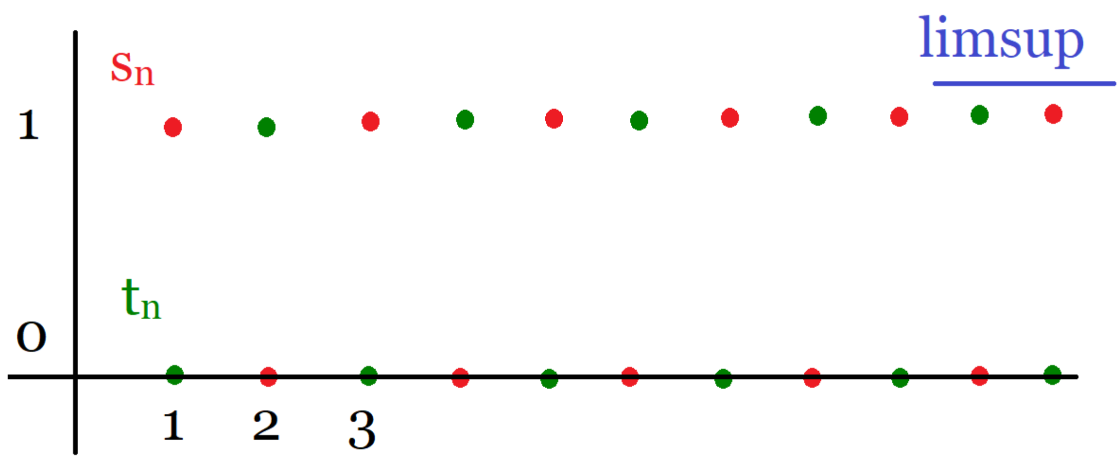
For the rest of today, we'll do some more practice with lim sup. First, let's prove a neat product rule for lim sup.

WARNING: In general

$$\limsup_{n\to\infty} s_n t_n \neq \left(\limsup_{n\to\infty} s_n\right)\left(\limsup_{n\to\infty} t_n\right)$$

Example:

$$s_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \qquad t_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$



Then $\limsup_{n\to\infty} s_n = 1$ and $\limsup_{n\to\infty} t_n = 1$. But for each n, either $s_n = 0$ or $t_n = 0$, so $s_n t_n = 0$, and therefore:

$$\limsup_{n\to\infty} s_n t_n = \limsup_{n\to\infty} 0 = 0 \neq 1 = \left(\limsup_{n\to\infty} s_n\right)\left(\limsup_{n\to\infty} t_n\right)$$

That said, not all is lost: Under some mild conditions of convergence, we can show that:

Fact:

If (s_n) converges to $s > 0$, and (t_n) is any sequence, then

$$\limsup_{n\to\infty} s_n t_n = \left(\limsup_{n\to\infty} s_n\right)\left(\limsup_{n\to\infty} t_n\right)$$

Note: It is important that $s > 0$ here. For $s \leq 0$ this property is False.

Proof:

STEP 1: Let's first show

$$\limsup_{n\to\infty} s_n t_n \geq \left(\limsup_{n\to\infty} s_n\right)\left(\limsup_{n\to\infty} t_n\right)$$

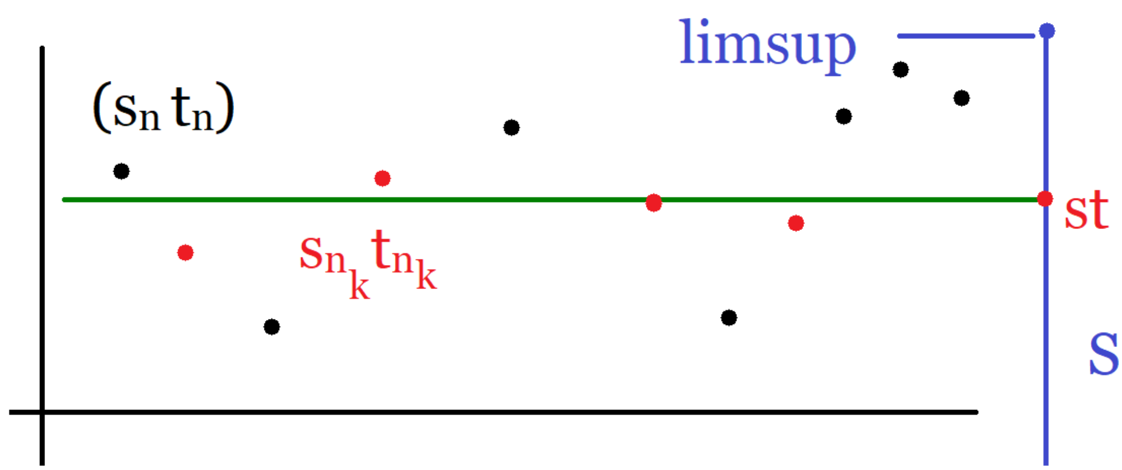
Let $t = \limsup_{n\to\infty} t_n$, which could be infinite!

Case 1: $t \in \mathbb{R}$ (t is finite)

Then, from last time, there is a subsequence (t_{n_k}) that converges to $\limsup_{n\to\infty} t_n = t$

But since (s_n) converges to s, the subsequence (s_{n_k}) (with the same n_k) converges to s as well

Therefore $\lim_{k\to\infty} s_{n_k} t_{n_k} = st$.



But then st is one possible limit point of $(s_n t_n)$, and since $\limsup_{n\to\infty} s_n t_n$ is the largest possible limit point of $(s_n t_n)$, we get:

$$\left(\limsup_{n\to\infty} s_n t_n\right) \geq st = \left(\limsup_{n\to\infty} s_n\right)\left(\limsup_{n\to\infty} t_n\right) \checkmark$$

We used that $\limsup_{n\to\infty} s_n = \lim_{n\to\infty} s_n = s$, since (s_n) converges to s

Case 2: $t = \infty$.

Then there is a subsequence (t_{n_k}) of (t_n) with $t_{n_k} \to \infty$.

And so $\lim_{k\to\infty} s_{n_k} t_{n_k} = s(\infty) = \infty$ (since $s > 0$) and therefore:

$$\limsup_{n\to\infty} s_n t_n \geq \lim_{k\to\infty} s_{n_k} t_{n_k} = \infty = \left(\limsup_{n\to\infty} t_n\right) \limsup_{n\to\infty} s_n \checkmark$$

Case 3: $t = -\infty$

Actually nothing to show, because

$$\left(\limsup_{n\to\infty} s_n\right)\left(\limsup_{n\to\infty} t_n\right) = s(-\infty) = -\infty$$

And therefore, since for any number x (even $\pm\infty$) we have $x \geq -\infty$, we have

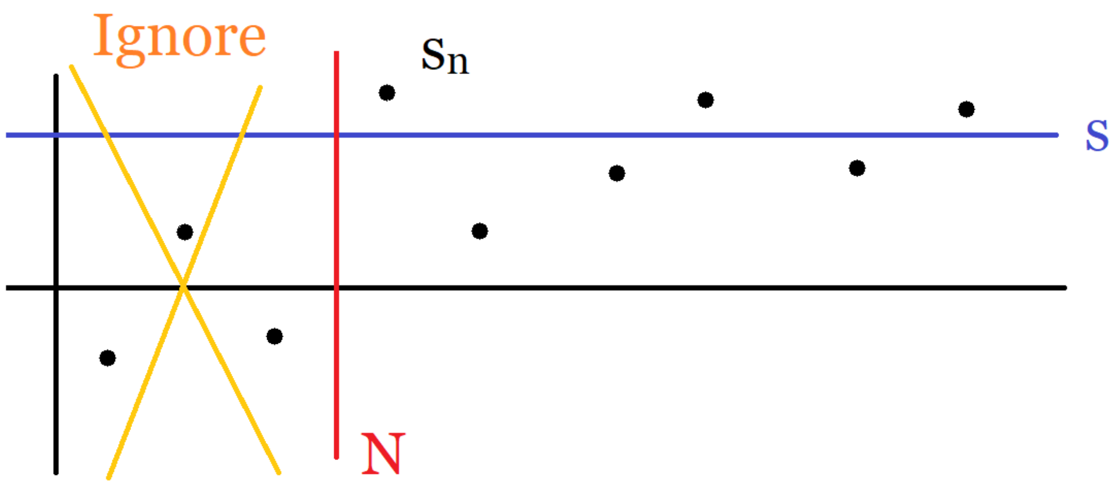
$$\limsup_{n\to\infty} s_n t_n \geq -\infty = \left(\limsup_{n\to\infty} s_n\right)\left(\limsup_{n\to\infty} t_n\right) \checkmark$$

STEP 2: Now let's show

$$\limsup_{n\to\infty} s_n t_n \leq \left(\limsup_{n\to\infty} s_n\right)\left(\limsup_{n\to\infty} t_n\right)$$

The amazing fact is that we can adapt our proof of **STEP 1** to prove this!

First of all, since $s_n \to s > 0$, there is some N such that, for all $n > N$, $s_n > 0$, so, ignoring the first few terms if necessary, assume WLOG that *for all* n, we have $s_n > 0$.



Then since $s_n \to s$ and $s_n \neq 0$, $\frac{1}{s_n} \to \frac{1}{s}$, so by **STEP 1**, we have:

$$\limsup_{n\to\infty} t_n = \limsup_{n\to\infty} \underbrace{\left(\frac{1}{s_n}\right)}_{\to\frac{1}{s}} s_n t_n \overset{\textbf{STEP 1}}{\geq} \frac{1}{s}\left(\limsup_{n\to\infty} s_n t_n\right)$$

And therefore:

$$\frac{1}{s}\left(\limsup_{n\to\infty} s_n t_n\right) \leq \limsup_{n\to\infty} t_n$$

That is:

$$\limsup_{n\to\infty} s_n t_n \leq s \limsup_{n\to\infty} t_n = \left(\limsup_{n\to\infty} s_n\right)\left(\limsup_{n\to\infty} t_n\right) \checkmark$$

So combining steps 1 and 2, we get:

$$\limsup_{n\to\infty} s_n t_n = s \limsup_{n\to\infty} t_n = \left(\limsup_{n\to\infty} s_n\right)\left(\limsup_{n\to\infty} t_n\right) \qquad \square$$

3. Pre-Ratio Test

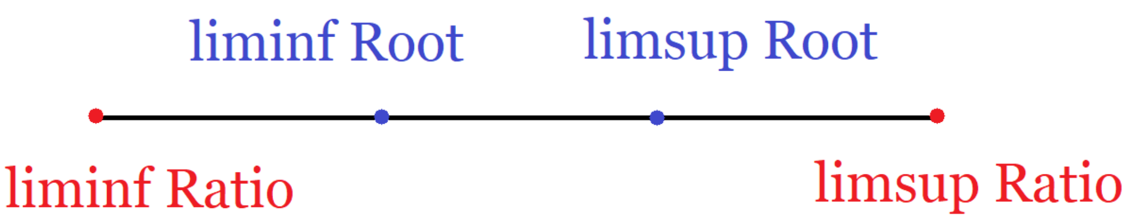
Video: Pre-Ratio Test

Finally, let's prove an identity that will be very useful in our discussion of the Ratio Test (in section 14)

Pre-Ratio Test:

If $s_n \neq 0$ for all n, then

$$\liminf_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| \leq \liminf_{n\to\infty}|s_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty}|s_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right|$$

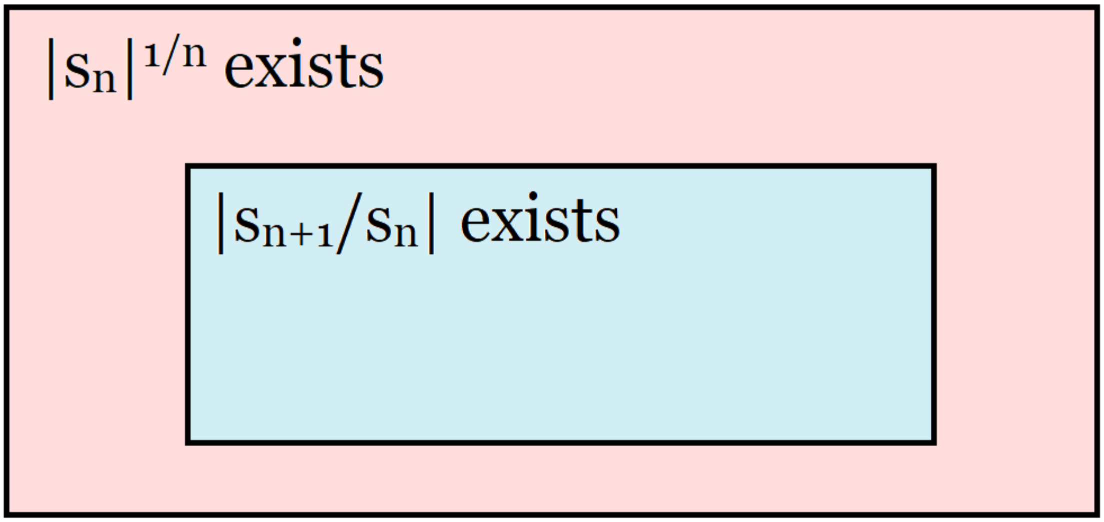


This inequality will show that the root test is strictly better than the ratio test. In fact, we have the following:

Corollary:

If $\lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| = L$, then $\lim_{n\to\infty}|s_n|^{\frac{1}{n}} = L$

So *if* the limit $\lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right|$ exists, then $\lim_{n\to\infty}|s_n|^{\frac{1}{n}}$ exists. But it *could* happen that $\lim_{n\to\infty}|s_n|^{\frac{1}{n}}$ exists but $\lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right|$ doesn't exist. That's why the root test is strictly better than the ratio test.



Proof of Corollary: Suppose $\lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| = L$, then:

$$L = \liminf_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| \leq \liminf_{n\to\infty}|s_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty}|s_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| = L$$

Therefore:

$$L \leq \liminf_{n\to\infty}|s_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty}|s_n|^{\frac{1}{n}} \leq L$$

Hence

$$\liminf_{n\to\infty}|s_n|^{\frac{1}{n}} = \limsup_{n\to\infty}|s_n|^{\frac{1}{n}} = L$$

And so, by the Limsup Squeeze Theorem

$$\lim_{n\to\infty}|s_n|^{\frac{1}{n}} = L \quad \square$$

Proof of Pre-Ratio Test:

Note: This proof is similar in spirit to Problem 12 in section 9, but just a bit fancier because we're using lim sup.

STEP 1: We want to show:

$$\liminf_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| \leq \liminf_{n\to\infty}|s_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty}|s_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right|$$

The middle inequality follows because lim inf $\leq$ lim sup and the first inequality can be proven similar to the third inequality, so let's just show that

$$\limsup_{n\to\infty}|s_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right|$$

$$\text{Let} \quad L = \limsup_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right|$$

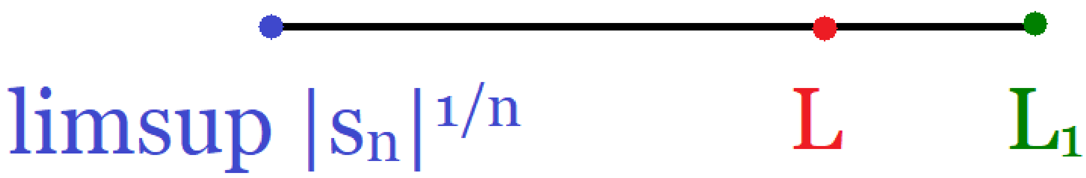
And we need to show that

$$\limsup_{n\to\infty}|s_n|^{\frac{1}{n}} \leq L$$

Note that the inequality is true if $L = \infty$, so from now on assume $L < \infty$.

Note: For reasons that will become apparent later, ideally we would like to have some space/wiggle room between the limsup and L. In order to get around that, notice that it's enough to show that:

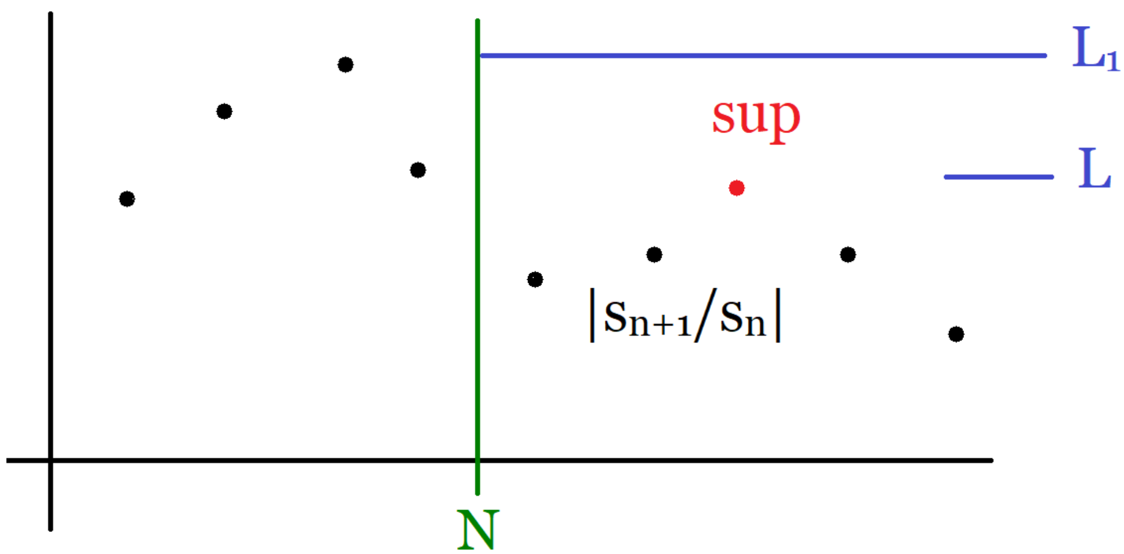
$$\limsup_{n\to\infty}|s_n|^{\frac{1}{n}} \leq L_1 \text{ for all } L_1 > L$$



This is *basically* the same thing as saying that if $a \leq b + \epsilon$ for all $\epsilon > 0$, then $a \leq b$

STEP 2: By definition of L and lim sup, we have

$$L = \limsup_{n\to\infty} \left| \frac{s_{n+1}}{s_n} \right| = \lim_{N\to\infty} \sup \left\{ \left| \frac{s_{n+1}}{s_n} \right| \mid n > N \right\}$$



But since $L < L_1$ by assumption, for N large enough, we have (see picture with sup above)

$$\sup \left\{ \left| \frac{s_{n+1}}{s_n} \right| \mid n > N \right\} < L_1$$

And so, by definition of sup, for all $n > N$,

$$\left|\frac{s_{n+1}}{s_n}\right| < L_1 \Rightarrow |s_{n+1}| < L_1 |s_n|$$

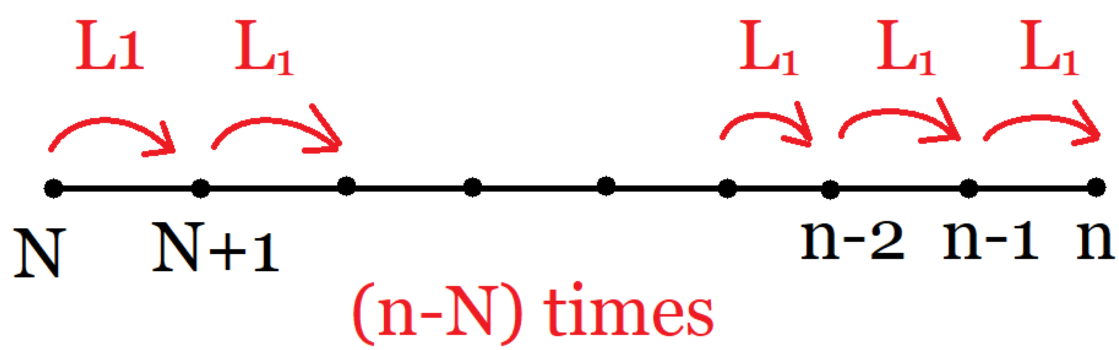
STEP 3: And so

$$|s_n| < L_1|s_{n-1}| < L_1 (L_1|s_{n-2}|) = (L_1)^2 |s_{n-2}| < (L_1)^3 |s_{n-3}| < \dots$$

And more generally:

Claim:

$$|s_n| < (L_1)^{n-N} |s_N| \text{ for all } n > N$$



This just follows from:

$$\begin{aligned} |s_n| &= \left(\frac{|s_n|}{|s_{n-1}|}\right)\left(\frac{|s_{n-1}|}{|s_{n-2}|}\right)\dots\left(\frac{|s_{N+1}|}{|s_N|}\right)|s_N| \\ &< \underbrace{(L_1)(L_1)\cdots(L_1)}_{n-N \text{ times}}|s_N| \\ &= (L_1)^{n-N}|s_N| \end{aligned}$$

STEP 4: But then, we get:

$$|s_n| < (L_1)^{n-N} |s_N| = (L_1)^n (L_1)^{-N} |s_N| = (L_1)^n a \Rightarrow |s_n| < (L_1)^n a$$

Where $a = (L_1)^{-N} |s_N| > 0$, and therefore:

$$|s_n|^{\frac{1}{n}} < ((L_1)^n a)^{\frac{1}{n}} = L_1 \left(a^{\frac{1}{n}}\right)$$

And finally, taking $\limsup$, we get:

$$\limsup_{n\to\infty} |s_n|^{\frac{1}{n}} \leq \limsup_{n\to\infty} L_1 \left(a^{\frac{1}{n}}\right) = \lim_{n\to\infty} L_1 \left(a^{\frac{1}{n}}\right) = L_1 \lim_{n\to\infty} \left(a^{\frac{1}{n}}\right) = L_1(1) = L_1$$

Where we used $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$ (Which is in section 9, see the Limit Example 9 video)

Therefore we conclude that:

$$\limsup_{n\to\infty} |s_n|^{\frac{1}{n}} \leq L_1$$

And since $L_1 \geq L$ was arbitrary, we get:

$$\limsup_{n\to\infty} |s_n|^{\frac{1}{n}} \leq L = \limsup_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| \quad \square$$