LECTURE 13: SERIES (I)

In the next *series* of lectures (pun intended), we'll talk about series, which are infinite sums of sequences.

1. Partial Sums

Video: Partial Sums

Goal: Given a sequence (a_n), what does it mean to take the sum of all the values of a_n

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots = ?$$

Intuitively: A series is just a really big sum, think $a_1 + a_2 + \dots a_{500}$

Example 1:

What would it mean for

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1?$$

(We will prove that formula in the next section)

The sum above is an *infinite* sum. Since we only know about *finite* sums, let's look at what are called the *partial sums*:

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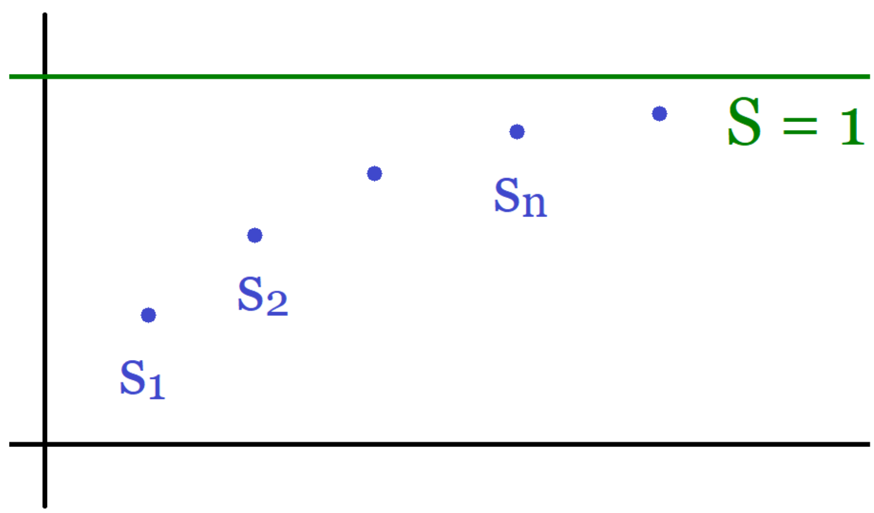
$$
\begin{aligned}
s_1 =& a_1 = \frac{1}{2} = 0.5 \\
s_2 =& a_1 + a_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 0.75 \\
s_3 =& a_1 + a_2 + a_3 = 0.875 \\
s_4 =& a_1 + a_2 + a_3 + a_4 = 0.9375 \\
s_5 \approx& 0.967 \\
s_6 \approx& 0.984 \\
s_7 \approx& 0.992
\end{aligned}
$$

In general:

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

In this example, the partial sums s_n converge to $S = 1$ as $n \to \infty$



And it's *this* limit S that we call $\sum_{n=1}^{\infty} a_n$:

Definition:

$$\sum_{n=1}^{\infty} a_n = S \text{ means } \lim_{n\to\infty} s_n = S \text{ , where}$$

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

Definition:

If the above limit exists, then we say $\sum a_n$ **converges**. Else, if $S = \pm\infty$ and/or the limit does not exist, then $\sum a_n$ **diverges**.

Example 2:

What is

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Look at:[1]

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} \\ &= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cdots + \cancel{\frac{1}{n}} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1} \\ &\to 1 \end{aligned}$$

Therefore, by definition,

[1] You can take $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ as a given, you don't need to show it, but usually you'd just use partial fractions from calculus

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Here is a useful class of convergent/divergent series that we'll use over and over again:

Fact (p-series):

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

Converges if and only if $p > 1$

Example 3: (1-series)

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$$

Example 4: (2-series)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots \text{ Converges}$$

Note: In fact, one can show that the value is $\frac{\pi^2}{6}$, see the following optional video if interested: Sum of $\frac{1}{n^2}$.

Note: This says something really interesting about numbers! There are much fewer integers that are squares (like $16 = 4^2$ or $49 = 7^2$) than there are integers. So few, in fact, that the $2-$series converges whereas the $1-$series diverges.

Finally, here's an **important** fact that is used over and over again in calculus (but that the book doesn't seem to mention)

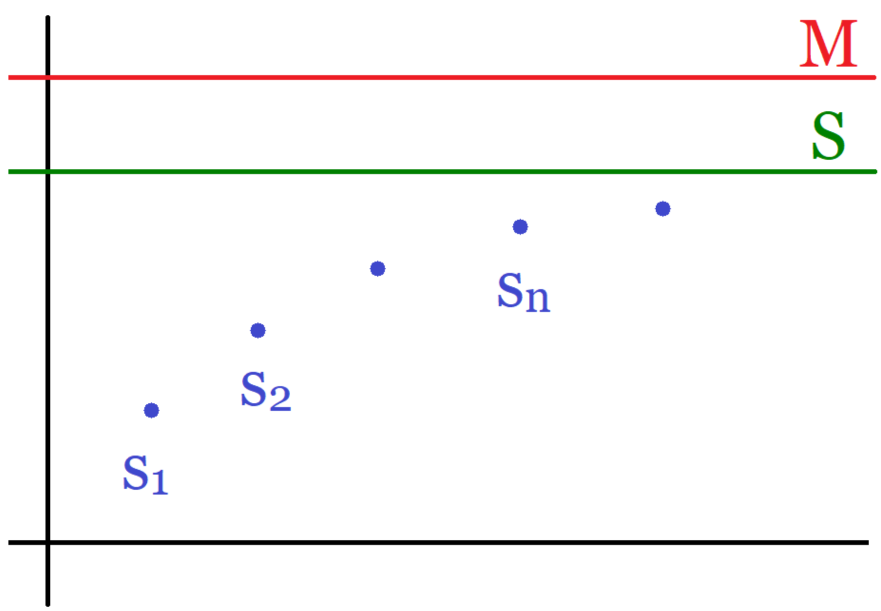
Fact:

Suppose $a_n \geq 0$ for all n

Then $\sum a_n$ converges if and only if (s_n) (as above) is **bounded**

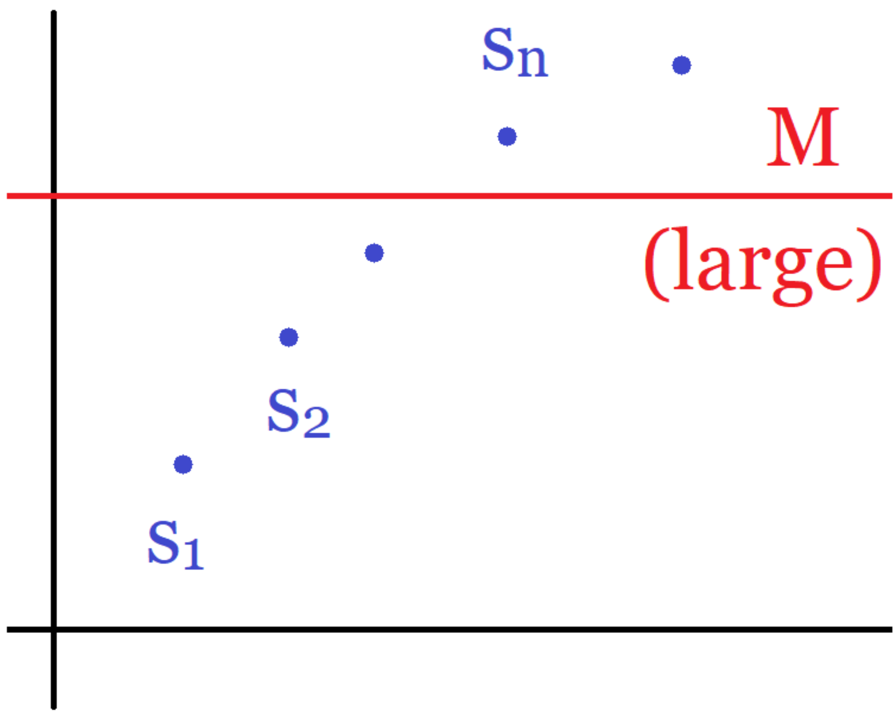
Note: This is what calculus textbooks mean when they say "A series converges if and only if it is bounded"

Why? Notice that, since $a_n \geq 0$, (s_n) is non-decreasing, at each step, you're just adding non-negative terms; compare with Example 1, where we had $s_1 = 0.5, s_2 = 0.75, s_3 = 0.875$.



($\Leftarrow$) (s_n) is bounded and non-decreasing, so by the Monotone Sequence Theorem, (s_n) converges, so by definition $\sum a_n$ converges

($\Rightarrow$) If (s_n) is not bounded, then, since (s_n) is non-decreasing, this implies $s_n \to \infty$, and therefore $\sum a_n = \infty$, which diverges $\square$



2. Geometric Series

Video: Geometric Series

In this section, we'll cover an important example of a series that's used over and over again in calculus and analysis, the *geometric series*:

Example 5:

$$\text{What is} \quad \sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \dots$$

Note: If $r \leq -1$ or $r \geq 1$, then $r^n \nrightarrow 0$, in which case $\sum r^n$ diverges (by the divergence test; see next section).

Therefore, from now on, consider only $-1 < r < 1$ (that is, $|r| < 1$).

Trick: Consider:

$$\begin{aligned} s_n =& \sum_{k=0}^{n} r^k = 1 + r + r^2 + \cdots + r^n \\ r s_n =& r\left(1 + r + r^2 + \dots r^n\right) \end{aligned}$$

$$\begin{aligned} s_n =& 1 + r + r^2 + \cdots + r^n \\ r s_n =& \quad\; r + r^2 + \cdots + r^n + r^{n+1} \end{aligned}$$

Therefore:

$$\begin{aligned} s_n - r s_n =& 1 + \cancel{r + r^2 + \cdots + r^n} - \cancel{\left(r + r^2 + \cdots + r^n\right)} - r^{n+1} \\ (1-r)s_n =& (1 - r^{n+1}) \\ s_n =& \frac{1 - r^{n+1}}{1-r} \end{aligned}$$

$$s_n = 1 + r + r^2 + \cdots + r^n = \left(\frac{1 - r^{n+1}}{1-r}\right)$$

Now since $-1 < r < 1$, then $r^{n+1} \to 0$ as $n \to \infty$, so

$$s_n = \left(\frac{1 - r^{n+1}}{1 - r}\right) \to \frac{1 - 0}{1 - r} = \frac{1}{1 - r}$$

And therefore, by definition:

Geometric Series:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}$$

This series converges if and only if $|r| < 1$

Example 6:

$$\text{Find} \quad \sum_{n=2}^{\infty} \left(\frac{-1}{3}\right)^n$$

First of all, the series converges since $\left|\frac{-1}{3}\right| = \frac{1}{3} < 1$.

Now, by the formula for the geometric series, we have:

$$\sum_{n=0}^{\infty} \left(\frac{-1}{3}\right)^n = 1 - \frac{1}{3} + \frac{1}{9} + \cdots = \frac{1}{1 - \left(-\frac{1}{3}\right)} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

And therefore:

$$\sum_{n=2}^{\infty} \left(\frac{-1}{3}\right)^n = \frac{1}{9} - \frac{1}{27} + \cdots = \frac{1}{9}\left(1 - \frac{1}{3} + \frac{1}{9} + \dots\right) = \frac{1}{9}\left(\frac{3}{4}\right) = \frac{1}{12}$$

3. The Cauchy Criterion

Video: The Cauchy Criterion

We would now like to prove some convergence tests for series, like the Divergence Test and Comparison Test (see below). In order to achieve this, we need to find a *better* way to define "$\sum a_n$ converges"

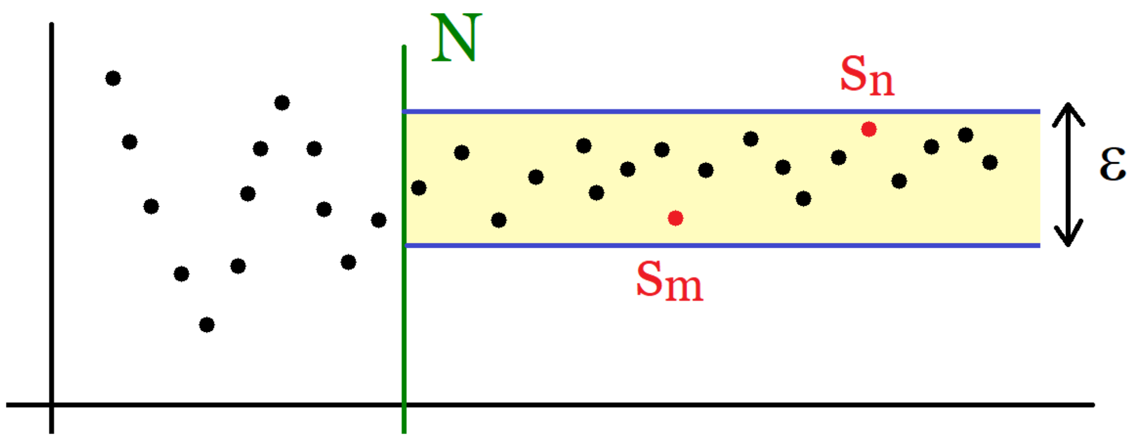
For this, let me remind you of the definition of Cauchy[2] sequences from section 10.

Recall:

A sequence (s_n) is **Cauchy** if, for all $\epsilon > 0$, there is N such that if $m, n > N$, then

$$|s_n - s_m| < \epsilon$$

Intuitively, this means that, after the threshold N, the terms (s_n) get closer and closer together.



Recall:

In $\mathbb{R}$, (s_n) converges $\Leftrightarrow$ (s_n) is Cauchy

[2] Voulez-vous Cauchy avec moi?

Which gives an alternative definition of convergence, in the case where your space is $\mathbb{R}$.

Let's tweak this definition in order to make it particularly attractive for series. For this, note the following:

(1) WLOG, assume $n > m$. This doesn't affect the result since $|s_n - s_m| = |s_m - s_n|$

(2) Replace m with $m - 1$ (ok since m is arbitrary)

(3) $n > m - 1 \Leftrightarrow n \geq m$ (for example, if $n > 8$, then $n \geq 9$)

Therefore, using the remarks above, the above definition becomes:

Definition:

(s_n) is Cauchy if, for all $\epsilon > 0$, there is N such that if $n \geq m > N$, then

$$|s_n - s_{m-1}| < \epsilon$$

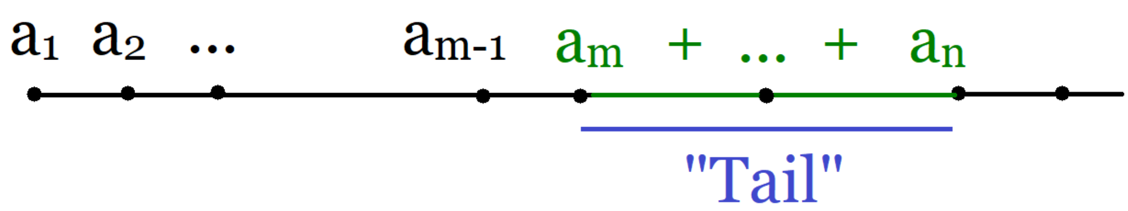
Important Application:

Suppose now that s_n is the sequence of partial sums, that is

$$s_n = \sum_{k=1}^{n} a_k = a_1 + \cdots + a_n$$

Then:

$$\begin{aligned} s_n - s_{m-1} &= (a_1 + \cdots + a_{m-1} + a_m + \cdots + a_n) - (a_1 + a_2 + \cdots + a_{m-1}) \\ &= a_m + \cdots + a_n \\ &= \sum_{k=m}^{n} a_k \text{ ("Tail of the series")} \end{aligned}$$



Therefore in the end, we get a powerful criterion for testing if a series converges, called the:

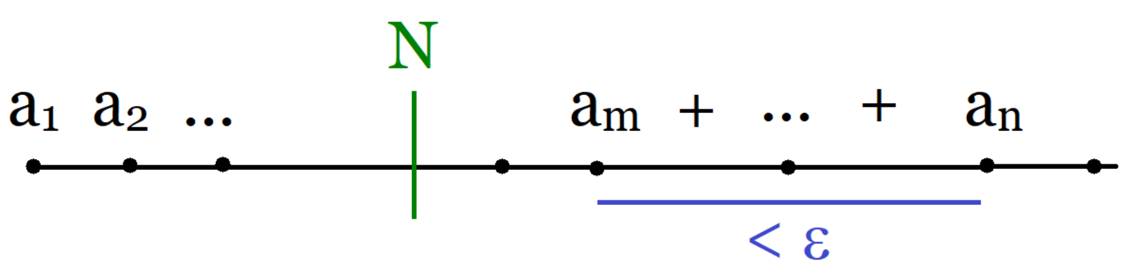
Cauchy Criterion:

A series $\sum_{n=1}^{\infty} a_n$ converges if and only if it satisfies the **Cauchy criterion**:

For all $\epsilon > 0$ there is N such that, if $n \geq m > N$, then

$$\left| \sum_{k=m}^{n} a_k \right| < \epsilon$$

In other words, no matter how small the error, the tail $\sum_{k=m}^{n} a_k$ of the series (no matter how long) eventually becomes as small as we want.



Example: Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, this means that (with $\epsilon = 0.0003$), no matter *what* tail $\sum_{k=m}^{n} \frac{1}{n^2}$ we pick, that tail will always be < 0.0003 provided m, n are large enough.

An example of such tail (with 3 terms) is:

$$\sum_{k=100}^{102} \frac{1}{k^2} = \frac{1}{(100)^2} + \frac{1}{(101)^2} + \frac{1}{(102)^2} \approx 0.000294 < 0.0003$$

But convergence means much more than that: Even with $5, 7$, or $10,000$ terms, the tail will be < 0.0003.

4. The Divergence Test

Using the Cauchy criterion, we can prove the Divergence Test:

The Divergence Test:

If the series $\sum a_n$ converges, then the sequence $a_n \to 0$

Or, equivalently, if $a_n \nrightarrow 0$, then $\sum a_n$ cannot converge.

This makes intuitively sense, because suppose for example that $a_n \to 2$, then in the series $\sum a_n$, we're eventually adding up terms that are close to 2, so $\sum a_n$ would eventually look like $2 + 2 + 2 + \cdots = 2$

Example 7:

Does the following series converge?

$$\sum_{n=1}^{\infty} \left(3 - \frac{2}{n}\right)$$

No because $3 - \frac{2}{n} \to 3 \neq 0$

Example 8:

Does the following series converge?

$$\sum_{n=1}^{\infty}(-1)^n = -1+1-1+1\dots$$

No because $\lim_{n\to\infty}(-1)^n$ does not exist, so it certainly does not converge to 0.

Proof: Let $\epsilon > 0$ be given, then, since $\sum a_n$ converges, by the Cauchy criterion, there is N such that if $n \geq m > N$, then

$$\left|\sum_{k=m}^{n} a_k\right| < \epsilon$$

Upshot: Since the above is true for *all* m and n (with $n \geq m > N$), it is *in particular* true for $n = m\,(> N)$

With the same N, if $m > N$, then you get:

$$\left|\sum_{k=m}^{m} a_k\right| = |a_m| < \epsilon$$

Therefore, for all $\epsilon > 0$ there is N such that if $m > N$, then $|a_m - 0| < \epsilon$

Therefore, by definition, $a_m \to 0$ as $m \to \infty$ $\square$

Notice how elegant this proof is! This is why we worked so hard to define the Cauchy criterion

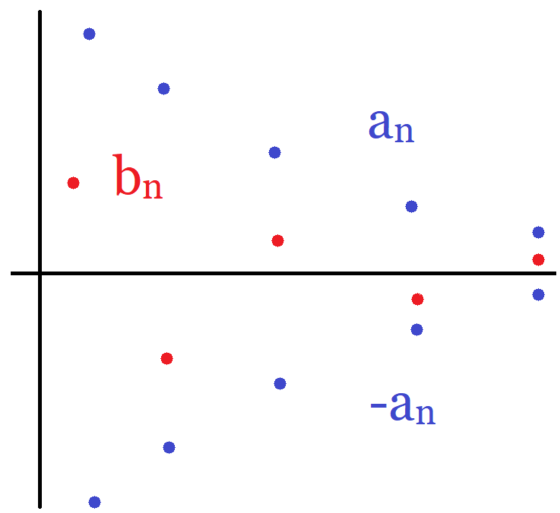
5. The Comparison Tests

Video: The Comparison Test

Let's now prove the comparison test(s). Intuitively, it says that if a series is less than a convergent series, then it converges.

Comparison Test 1:

Suppose $a_n \geq 0$ for all n. If $|b_n| \leq a_n$ for all n and $\sum a_n$ converges, then $\sum b_n$ converges



It's kind of like a squeeze theorem, but for series, since b_n is squeezed between $-a_n$ and a_n

Intuitively, think of it as: If $b_n \leq a_n$ and $\sum a_n < \infty$, then $\sum b_n < \infty$

Example 9:

Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

Let $b_n = \frac{1}{n^2+1}$ then:

$$|b_n| = \frac{1}{n^2+1} \leq \frac{1}{n^2} =: a_n$$

But since $\sum a_n = \sum \frac{1}{n^2}$ converges, then $\sum b_n = \sum \frac{1}{n^2+1}$ converges.

Proof: Let $\epsilon > 0$ be given. Then, by the Cauchy criterion for $\sum a_n$, there is N such that if $n \geq m > N$, then

$$\left| \sum_{k=m}^{n} a_k \right| = \sum_{k=m}^{n} a_k < \epsilon$$

(Here we used that $a_k \geq 0$ by assumption)

But then, with the same N, if $n \geq m > N$, then

$$\left| \sum_{k=m}^{n} b_k \right| \leq \sum_{k=m}^{n} |b_k| \leq \sum_{k=m}^{n} a_k < \epsilon$$

Where we used the triangle inequality and the fact that $|b_k| \leq a_k$.

Therefore, by the Cauchy criterion for (b_n), $\sum b_n$ converges $\square$

Here's a neat application of the convergence test:

Definition:

$\sum a_n$ converges **absolutely** if $\sum |a_n|$ converges

Example 10:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} \dots$$

Converges absolutely since

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots \text{ converges}$$

Corollary:

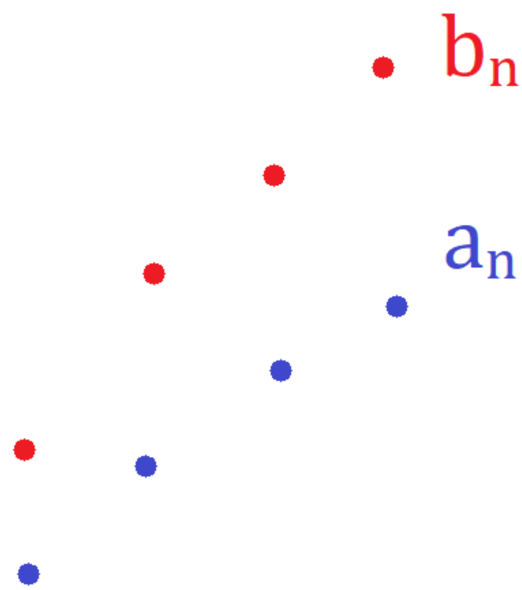
Absolutely convergent series converge

Proof: This just follows because $|a_n| \leq |a_n|$ so since $\sum |a_n|$ converges, by comparison, $\sum a_n$ converges as well. □

There's also a simple analog of the comparison test, but for divergent series: If a series is larger than one that goes to ∞, then that series also goes to ∞:

Comparison Test 2:

Suppose $b_n \geq a_n$ for all n and $\sum_n a_n = \infty$, then $\sum_n b_n = \infty$



Proof: Let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$ be the partial sums of (a_n) and (b_n) respectively.

Then, since by assumption $b_k \geq a_k$ for all k, we get

$$\sum_{k=1}^{n} b_k \geq \sum_{k=1}^{n} a_k \Rightarrow t_n \geq s_n$$

Since $\sum a_n = \infty$, by definition we must have $s_n \to \infty$.

Therefore, by comparison (of limits), we must have $t_n \to \infty$, that is $\sum b_n \to \infty$. $\square$

Example 11:

Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{n}{n^2+3}$$

Note: This looks a lot like $\sum \frac{n}{n^2} = \sum \frac{1}{n}$ which diverges.

Notice that $3 \leq 3n^2$, therefore:

$$b_n =: \frac{n}{n^2+3} \geq \frac{n}{n^2+3n^2} = \frac{n}{4n^2} = \frac{1}{4n} =: a_n$$

Hence $b_n \geq a_n$ for all n. But since

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{4n} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

By comparison we get

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{n}{n^2+3} = \infty$$