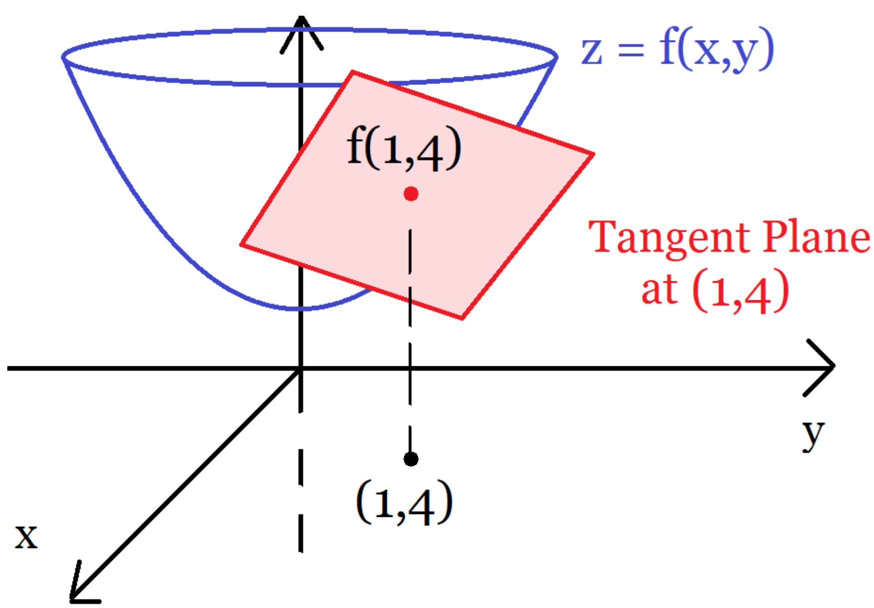
LECTURE 13: TANGENT PLANES

1. Tangent Planes

Video: Tangent Planes

Just like derivatives lead to tangent lines, partial derivatives lead to tangent planes.



Example 1:

Find the equation of the tangent plane of $f(x,y) = \sqrt{xy}$ at $(1,4)$

Date: Monday, September 27, 2021.

Tangent Plane

The equation of the tangent plane of f at $(1,4)$ is:

$$z - f(1,4) = f_x(1,4)(x-1) + f_y(1,4)(y-4)$$

(Just the point-slope formula, but with two variables. Intuitively, this makes sense if you think of f_x and f_y as the slopes in the x and y directions)

$$f(1,4) = \sqrt{(1)(4)} = 2$$

$$f_x(x,y) = (\sqrt{xy})_x = \left(\frac{1}{2\sqrt{xy}}\right) y \Rightarrow f_x(1,4) = \left(\frac{1}{2\sqrt{4}}\right) 4 = 1$$

$$f_y(x,y) = (\sqrt{xy})_y = \left(\frac{1}{2\sqrt{xy}}\right) x \Rightarrow f_y(1,4) = \left(\frac{1}{2\sqrt{4}}\right) 1 = \frac{1}{4}$$

Answer: $z - 2 = 1(x-1) + \frac{1}{4}(y-4)$

2. Linear Approximations

Video: Linear Approximations and Differentials

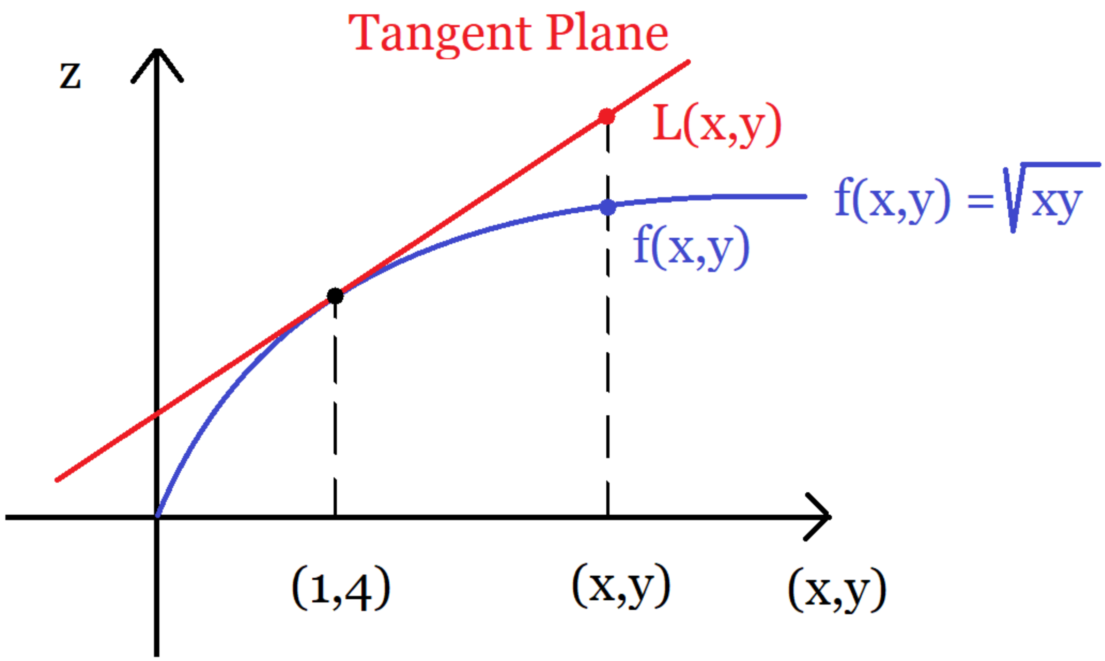
Why care about tangent planes? Because they allow us to approximate complicated values of f

Example 2:

Let $f(x,y) = \sqrt{xy}$

(a) Find the linear approximation $L(x,y)$ to f at $(1,4)$

Profile View:



Linear Approximation:

$$L(x,y) = \text{Equation of tangent plane at } (1,4)$$
$$= f(1,4) + f_x(1,4)(x-1) + f_y(1,4)(y-4)$$

(Compare to $L(x) = f(a) + f'(a)(x-a)$ in Calculus I)

Hence, using the previous example, we get:

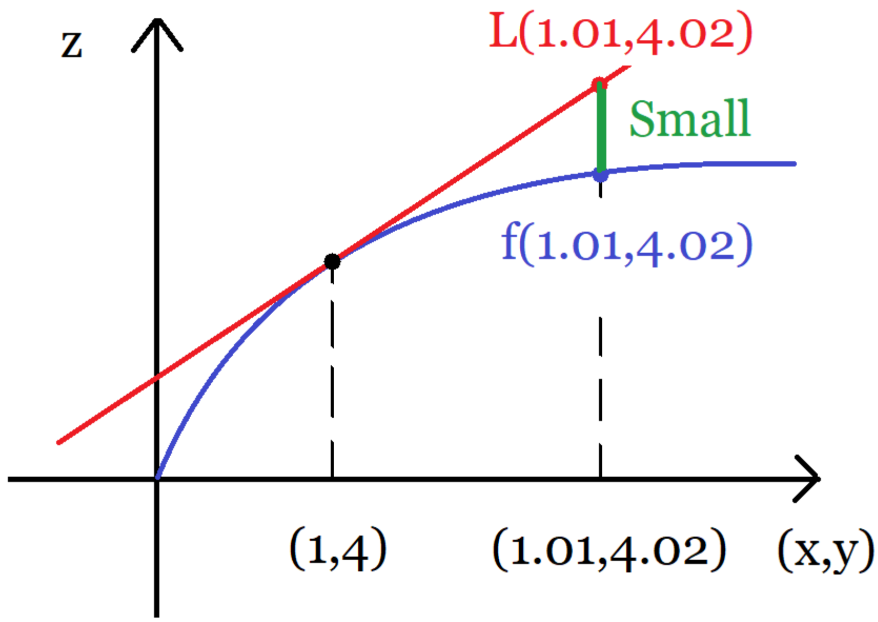
$$L(x,y) = 2 + (x-1) + \frac{1}{4}(y-4)$$

(b) Use $L(x,y)$ to approximate $\sqrt{(1.01)(4.02)}$

Point: Near $(1,4)$, the tangent plane is a *good* approximation to f

Fact:

For (x,y) near $(1,4)$, we have $L(x,y) \approx f(x,y)$



$$\begin{aligned}
\sqrt{(1.01)(4.04)} =& f(1.01, 4.02) \\
\approx& L(1.01, 4.02) \quad \text{(By Fact)} \\
=& 2 + (1.01 - 1) + \frac{1}{4}(4.02 - 4) \quad \text{(By (a))} \\
=& 2 + 0.01 + \frac{1}{4}(0.02) \\
=& 2 + 0.01 + 0.005 \\
=& 2.015
\end{aligned}$$

Note: Compare to the actual value of 2.014994...

Example 3:

Approximate $\ln(0.97)e^{0.02}$

(1)

$$f(x,y) = \ln(x)e^y \qquad \text{Point: } (1,0)$$

(2)

$$L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$$

(3)

$$\begin{aligned} f(1,0) &= \ln(1)e^0 = 0 \\ f_x(x,y) &= \left(\frac{1}{x}\right)e^y \Rightarrow f_x(1,0) = \left(\frac{1}{1}\right)e^0 = 1 \\ f_y(x,y) &= \ln(x)e^y \Rightarrow f_y(1,0) = \ln(1)e^0 = 0 \end{aligned}$$

Therefore:

$$L(x,y) = 0 + 1(x-1) + 0(y-0) = x - 1$$

(4) Hence:

$$\begin{aligned} \ln(0.97)e^{0.02} &= f(0.97, 0.02) \\ &\approx L(0.97, 0.02) \\ &= 0.97 - 1 \\ &= -0.03 \end{aligned}$$

(Compare to the Actual Value of $-0.0311\cdots$)

3. Differentials

Now let's talk about differentials, which is just the different side of the same coin!

Example 4:

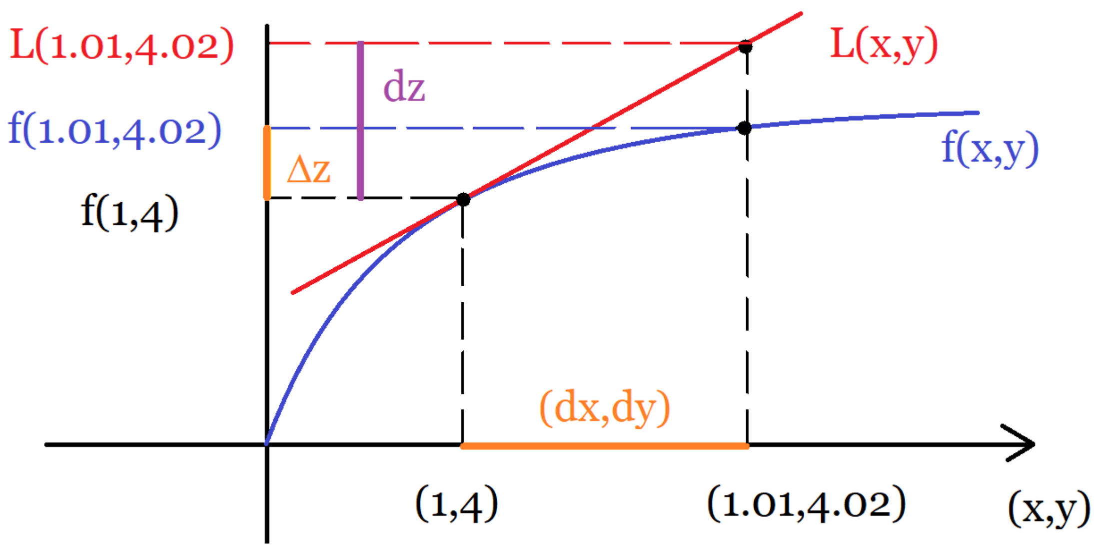
Use **differentials** to approximate $\sqrt{(1.01)(4.02)}$

(Same example as above, just to show you that we get the same answer)

$f(x, y) = \sqrt{xy}$, Point $= (1, 4)$

It's *literally* the same thing as linear approximations, but with different notation. Here instead of talking about closeness, we talk about small errors.

Profile View:



Notation

$$\begin{aligned} dx =& \Delta x = 1.01 - 1 = 0.01 \quad \text{(Error in } x\text{)} \\ dy =& \Delta y = 4.02 - 4 = 0.02 \quad \text{(Error in } y\text{)} \\ \Delta z =& f(1.01, 4.02) - f(1, 4) = \sqrt{(1.01)(4.02)} - 2 \\ & \text{(Actual Error in } z\text{)} \end{aligned}$$

Definition

$$dz = f_x(1,4)dx + f_y(1,4)dy \quad \text{(Calculus Error)}$$

(Compare to $dy = f'(x)dx$ in Calculus I, it's the error by using L instead of f)

$$dz = 1(0.01) + \frac{1}{4}(0.02) = 0.015$$

Fact:

For (x, y) near $(1, 4)$, we have $\Delta z \approx dz$

Therefore:

$$\begin{aligned} \Delta z \approx& dz \\ \sqrt{(1.01)(4.02)} - 2 \approx& 0.015 \\ \sqrt{(1.01)(4.02)} \approx& 2.015 \qquad \text{(Same answer as above)} \end{aligned}$$

Example 5:

Use differentials to approximate $\ln(0.97)e^{0.02}$

$$f(x, y) = \ln(x)e^y \qquad \text{Point: } (1, 0)$$

$$\begin{aligned} dz =& f_x(1,0)dx + f_y(1,0)dy \\ =& 1(0.97-1) + 0(0.02-0) \text{ (From above)} \\ =& -0.03 \end{aligned}$$

Then $\Delta z \approx dz$ gives:

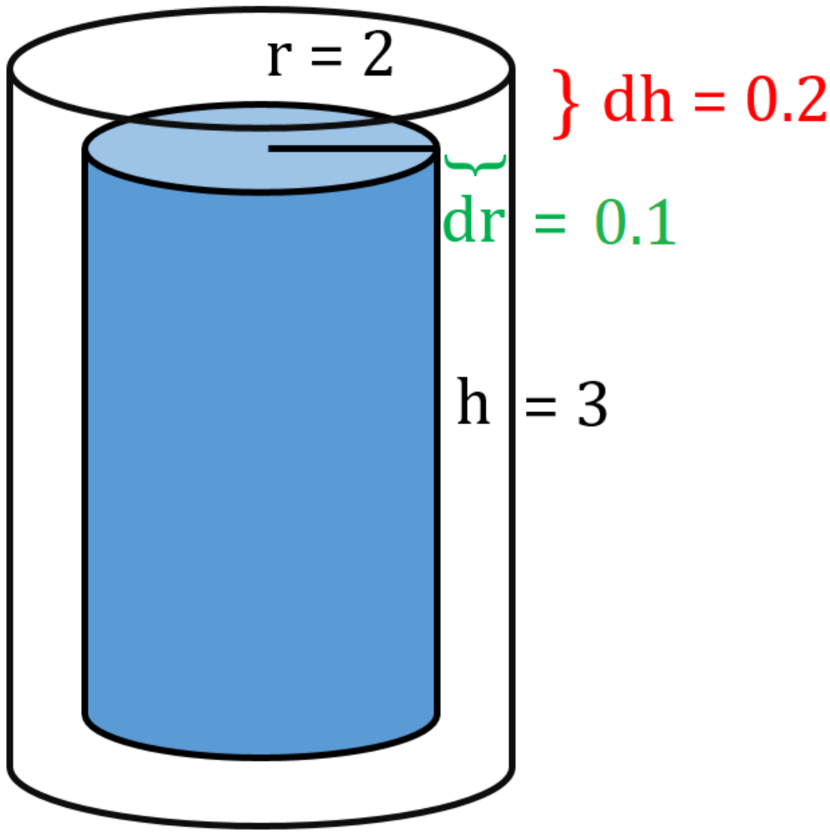
$$\begin{aligned} f(0.97, 0.02) - f(1,0) &\approx -0.03 \\ \ln(0.97)e^{0.02} - \underbrace{\ln(1)e^{0}}_{0} &\approx -0.03 \\ \ln(0.97)e^{0.02} &\approx -0.03 \end{aligned}$$

4. Application

As an application, let's try to measure the error in calculating the volume of a cylinder. This is very useful in engineering, where calculations are rarely exact.

Example 6:

Estimate the error in calculating the volume of a cylinder with radius $r = 2$ and height $h = 3$, where the error in measuring the radius is $dr = 0.1$ and the error in measuring the height is $dh = 0.2$



STEP 1: Prep Work

$$\begin{aligned} V &= V(r,h) = \pi r^2 h \\ r &= 2, \quad dr = 0.1 \\ h &= 3, \quad dh = 0.2 \end{aligned}$$

STEP 2:

$$\begin{aligned} \text{Actual Error } &= \Delta V \\ \text{Calculus Error } &= dV = V_r(2,3)dr + V_h(2,3)dh \end{aligned}$$

$$\begin{aligned} V_r(r,h) &= \pi(2r)h = 2\pi rh \\ V_r(2,3) &= 2\pi(2)(3) = 12\pi \\ V_h(r,h) &= \pi r^2 \\ V_h(2,3) &= \pi(2)^2 = 4\pi \end{aligned}$$

Therefore:

$$dV = (12\pi)(0.1) + (4\pi)(0.2) = 1.2\pi + 0.8\pi = 2\pi$$

(3) The error in measurement is $\Delta V \approx dV = 2\pi$.

This tells you that if the measures of r and h are off by 0.1 and 0.2, the measure of the volume is off by about 2π, which is huge!